Lecture 2

(2.1) Drinfeld's presentation

$Y := Y_\hbar(\mathfrak{sl}_2)$ is the unital associative algebra (over $\mathbb{C}[\hbar]$ or $\mathbb{C}$ with $\hbar \in \mathbb{C}^\times$) generated by

$$\xi_r,\ x_r^\pm \qquad r \in \mathbb{Z}_{\geq 0}$$

subject to the following relations:

(Y1) $\quad [\xi_r, \xi_s] = 0$

(Y2) $\quad [\xi_0, x_r^\pm] = \pm 2 x_r^\pm$

(Y3) $\quad [\xi_{r+1}, x_s^\pm] - [\xi_r, x_{s+1}^\pm] =$

$$= \pm \hbar \left(\xi_r \cdot x_s^\pm + x_s^\pm \cdot \xi_r \right)$$

(Y4) $\quad [x_{r+1}^\pm, x_s^\pm] - [x_r^\pm, x_{s+1}^\pm] =$

$$= \pm \hbar \left(x_r^\pm \cdot x_s^\pm + x_s^\pm \cdot x_r^\pm \right)$$

(Y5) $\quad [x_r^+, x_s^-] = \xi_{r+s}$

Remarks

(1) $\hbar$ formal : Y is $\mathbb{N}$-graded with $\deg \hbar = 1$ and

$$\deg \xi_r = r = \deg x_r^{\pm}$$

$\hbar \in \mathbb{C}^\times$: Y is $\mathbb{N}$-filtered

(2) $Y/_{\hbar} Y \simeq U(\mathfrak{g}[t])$ $\quad \begin{cases} x_r^+ \equiv e \otimes t^r \\ \xi_r \equiv h \otimes t^r \\ x_r^- \equiv f \otimes t^r \end{cases}$

(2.2) Functional relations

Generating series :

$$\xi(u) := 1 + \hbar \sum_{r \geq 0} \xi_r u^{-r-1}$$

$$x^{\pm}(u) := \hbar \sum_{r \geq 0} x_r^{\pm} u^{-r-1}$$

Lemma Y is generated by $x^{\pm}(u)$, $\xi(u)$ with relations

(Y1) $[\xi(u), \xi(v)] = 0$

(Y2) + (Y3) $\xi(u) \cdot x^{\pm}(v) \cdot \xi(u)^{-1} =$

$$= \frac{u - v \pm \hbar}{u - v \mp \hbar} x^{\pm}(v) \mp \frac{2\hbar}{u - v \mp \hbar} x^{\pm}(u \mp \hbar)$$

(Y4) $$x^\pm(u)\cdot x^\pm(v) - \frac{u-v\pm\hbar}{u-v\mp\hbar}\, x^\pm(v)\, x^\pm(u) =$$

$$= \frac{\hbar}{u-v\mp\hbar}\left(\left[x_0^\pm, x^\pm(u)\right] - \left[x^\pm(v), x_0^\pm\right]\right)$$

(Y5) $$\left[x^+(u), x^-(v)\right] = \frac{\hbar}{u-v}\left(\xi(v)-\xi(u)\right)$$

Remark Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra.
The Yangian $Y_\hbar(\mathfrak{g})$ is generated by $x_i^\pm(u)$, $\xi_i(u)$
with relations (Y1) - (Y5) with $\hbar \rightsquigarrow \hbar\cdot d_i a_{ij}/2$
plus the Drinfeld - Serre relations ($i\neq j$, $m = 1-a_{ij}$)

$$\sum_{\pi\in S_m}\left[x_i^\pm(u_{\pi(1)}), \left[x_i^\pm(u_{\pi(2)}), \cdots \right.\right.$$

$$\left.\left.\cdots \left[x_i^\pm(u_{\pi(m)}), x_j^\pm(v)\right]\cdots\right]\right] = 0$$

pf (Y1) Clear.

(Y5) Note that

$$\xi(v) - \xi(u) = \hbar \sum_{r \geq 0} \xi_r v^{-s-1} - \hbar \sum_{s \geq 0} \xi_s u^{-r-1}$$

$$(u-v) \cdot [x^+(u), x^-(v)] =$$

$$= \hbar^2 \sum_{r,s \geq 0} [x_r^+, x_s^-] \left(u^{-r} \cdot v^{-s-1} - u^{-r-1} \cdot v^{-s} \right)$$

$$= \hbar^2 \sum_{t \geq 0} \xi_t \underbrace{\sum_{r+s=t} \left(u^{-r} \cdot v^{-s-1} - u^{-r-1} \cdot v^{-s} \right)}_{\left(v^{-t-1} - u^{-t-1}\right)}$$

(Y2) + (Y3) We want to show that

(*) $(u - v \mp \hbar) \cdot \xi(u) \cdot x^{\pm}(v) - (u - v \pm \hbar) \cdot x^{\pm}(v) \cdot \xi(u)$

is independent of v:

(uv^{-s-1}) $\quad \hbar (x_s^{\pm} - x_s^{\pm}) = 0$

(v^{-s-1}) $\quad \hbar^2 [\xi_0, x_s^{\pm}] \mp 2 \hbar \hbar x_s^{\pm} = 0$ iff (Y2)

$(u^{-r-1} \cdot v^{-s-1})$ $\quad \hbar^2 \Big([\xi_{r+1}, x_s^{\pm}] - [\xi_r, x_{s+1}^{\pm}]$

$\qquad \mp \hbar \left(\xi_r \cdot x_s^{\pm} + x_s^{\pm} \cdot \xi_r \right) \Big) = 0$

iff (Y3)

Being independent of v, $(*)$ is equal to its value at $v = u \mp \hbar$, thus

$$(u - v \mp \hbar) \cdot \xi(u) \cdot x^{\pm}(v) - (u - v \pm \hbar) x^{\pm}(v) \cdot \xi(u)$$

$$= \mp 2\hbar \cdot x^{\pm}(u \mp \hbar) \cdot \xi(u)$$

(Y4)

$$\hbar^2 \sum_{r,s \geq 0} \left([x^{\pm}_{r+1}, x^{\pm}_{s}] - [x^{\pm}_{r}, x^{\pm}_{s+1}] \right) u^{-r-1} \cdot v^{-s-1} =$$

$$= \pm \hbar^2 \hbar \sum_{r,s \geq 0} \left(x^{\pm}_{r} \cdot x^{\pm}_{s} + x^{\pm}_{s} \cdot x^{\pm}_{r} \right) u^{-r-1} \cdot v^{-s-1}$$

note that $\hbar \sum_{r \geq 0} x^{\pm}_{r+1} \cdot u^{-r-1} = u \cdot x^{\pm}(u) - \hbar x^{\pm}_0$,

thus we get

$$\left[u x^{\pm}(u) - \hbar x^{\pm}_0 , x^{\pm}(v) \right] - \left[x^{\pm}(u), v x^{\pm}(v) - \hbar x^{\pm}_0 \right] =$$

$$= \pm \hbar \left(x^{\pm}(u) x^{\pm}(v) + x^{\pm}(v) x^{\pm}(u) \right)$$

and

$$(u - v \pm \hbar) x^{\pm}(u) x^{\pm}(v) - (u - v \mp \hbar) x^{\pm}(v) x^{\pm}(u) =$$

$$= \hbar \left([x^{\pm}_0, x^{\pm}(v)] - [x^{\pm}(u), x^{\pm}_0] \right)$$

$\square$

(2.3) Shift automorphism

Relying on the "functional presentation", we see immediately that there is a 1-parameter family of automorphisms $\tau_s : Y \longrightarrow Y$, $s \in \mathbb{C}$, given by

$$\tau_s(y(u)) = y(u-s) \qquad y = x^\pm, \xi$$

(2.4) Levendorskii's presentation

Y' is generated by $t_0, t_1, x_0^\pm, x_1^\pm$ with relations

$r, s \in \{0,1\}$

(L1) $[t_r, t_s] = 0$

(L2) $[t_0, x_r^\pm] = \pm 2 x_r^\pm$

(L3) $[t_1, x_0^\pm] = \pm 2 x_1^\pm$

(L4) $[x_1^\pm, x_0^\pm] - [x_0^\pm, x_1^\pm] = \pm 2\hbar (x_0^\pm)^2$

(L5)
$$[x_0^+, x_0^-] = t_0$$
$$[x_1^+, x_0^-] = t_1 + \frac{\hbar}{2} t_0^2 = [x_0^+, x_1^-]$$

(L7) $[[t_1, x_1^+], x_1^-] + [x_1^+, [t_1, x_1^-]] = 0$

Remark Let $\mathfrak{g}$ be f.d. simple Lie algebra. Then,

$Y'_\hbar(\mathfrak{g})$ is generated by $t_{i,0}$, $t_{i,1}$, $x^\pm_{i,0}$, $x^\pm_{i,1}$

with relations L1 – L7 above with $2 \rightsquigarrow d_i a_{ij}$,

$\hbar \rightsquigarrow \hbar\, d_i a_{ij}/2$, δ_{ij} in r.h.s. of L5, and

$(x_0^\pm)^2 \rightsquigarrow (x^\pm_{i,0} x^\pm_{j,0} + x^\pm_{j,0} x^\pm_{i,0})$ in r.h.s. of L4,

together with the Serre relations

$$(\text{L6}) \qquad (\operatorname{ad} x^\pm_{i,0})^{1-a_{ij}} (x^\pm_{j,0}) = 0$$

Thm (Levendorskii 94) $\quad Y' \xrightarrow{\sim} Y$

$$\begin{array}{ccc} x_0^\pm,\ x_1^\pm & & x_0^\pm,\ x_1^\pm \\ & \rightsquigarrow & \\ t_0,\ t_1 & & \xi_0,\ \xi_1 - \frac{\hbar}{2}\xi_0^2 \end{array}$$

Remark (1) $\hbar \sum_{r \geq 0} t_r u^{-r-1} := \log(\xi(u))$

(2) Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra.

A similar presentation of $Y_\hbar(\mathfrak{g}')$ was obtained

by Guay, Nakajima, Wendlandt (2018) given

$\det(A_{\bar{i}\bar{j}}) \neq 0$ for any $i \neq j$ and $\exists\, a_{\bar{i}\bar{j}} = -1$.

(e.g. $A \neq A_1^{(1)}, A_2^{(2)}$)

(2.4.1) Proof of Levendorskii's presentation

One checks easily that $\mathcal{Y}' \longrightarrow \mathcal{Y}$ is an algebra map.

In order to define the converse $\rho : \mathcal{Y} \longrightarrow \mathcal{Y}'$ we set $\rho(x_r^\pm) = x_r^\pm$ for $r = 0, 1$, $\rho(\xi_0) = t_0$, $\rho(\xi_1) = t_1 + \hbar/2\, t_0^2$, we define recursively

$$2\rho(x_{r+1}^\pm) = \pm [t_1, \rho(x_r^\pm)]$$

and

$$\rho(\xi_r) = [\rho(x_r^+), x_0^-]$$

$\square$

(2.5) Rationality property

Lemma On a f.d. representation (V, π) of Y the formal series $\pi(\xi(u))$, $\pi(x^{\pm}(u))$ in $\mathrm{end}(V)[\![u^{-1}]\!]$ are Taylor series expansions of $\mathrm{end}(V)$-valued rational functions in u.

pf We have $\mathrm{ad}(t_1)(x_r^{\pm}) = \pm 2 x_{r+1}^{\pm}$, thus

$$\pm \frac{1}{2} \mathrm{ad}(t_1) x^{\pm}(u) = u \cdot x^{\pm}(u) - \hbar x_0^{\pm}$$

and

$$\left(u \mp \frac{1}{2} \mathrm{ad}(t_1)\right)\left(x^{\pm}(u)\right) = \hbar x_0^{\pm}$$

$$\Rightarrow \quad x^{\pm}(u) = \left(u \mp \frac{1}{2} \mathrm{ad}(t_1)\right)^{-1} \hbar x_0^{\pm}$$

V f.d $\rightsquigarrow$ $\mathrm{end}(V)$ f.d. $\circlearrowleft$ $\mathrm{ad}(t_1)$

$\rightsquigarrow$ $\left(u \mp \frac{1}{2}\mathrm{ad}(t_1)\right)^{-1}$ is a rational $\mathrm{end}(\mathrm{end}(V))$-valued function in u

Finally, $\xi(u) = 1 + [x^+(u), x_0^-]$. $\square$

(2.6) The coproduct on $Y = Y_\hbar(\mathfrak{sl}_2)$

There is a unique algebra map $\Delta : Y \to Y \otimes Y$ given by

$$\Delta(\xi_0) = \xi_0 \otimes 1 + 1 \otimes \xi_0$$

$$(*) \qquad \Delta(x_0^\pm) = x_0^\pm \otimes 1 + 1 \otimes x_0^\pm$$

$$\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1 - 2\hbar\, x_0^- \otimes x_0^+$$

Recall that by (L3) $x_1^\pm = \pm \frac{1}{2}[t_1, x_0^\pm]$.

Rmk (i) $U\mathfrak{sl}_2 \subseteq Y_\hbar \mathfrak{sl}_2$ is a Hopf subalgebra.

(ii) $U\mathfrak{sl}_2$ and t_1 generate Y, thus $(*)$ determines Δ uniquely. However, proving that Δ is an algebra map is non-trivial and relies on Levendorskii's presentation.

(iii) Let $\mathfrak{g}$ be a f.d. simple Lie algebra.

$Y_\hbar(\mathfrak{g})$ is generated by $U\mathfrak{g} \subseteq Y_\hbar(\mathfrak{g})$ and $t_{i,1}$ $(i \in I)$. Let Φ be the set of roots of $\mathfrak{g}$.

For any $\beta \in \Phi^+$, we choose dual root vectors $x_{\beta,0}^\pm$ in $U\mathfrak{g} \subseteq Y_\hbar(\mathfrak{g})$.

In this case, Δ is given by

$$\Delta(y_0) = y_0 \otimes 1 + 1 \otimes y_0 \qquad y = \xi_i, x_i^{\pm}$$

$$\Delta(t_{i,1}) = t_{i,1} \otimes 1 + 1 \otimes t_{i,1} - \hbar \sum_{\beta > 0} (\beta, \alpha_i) x_{\beta,0}^{-} \otimes x_{\beta,0}^{+}$$

(iv) The coproduct of $Y_\hbar(\mathfrak{g})$ originally appeared in Drinfeld's J-presentation, relying on the Casimir tensor $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$. No explicit formula for $\Delta(x_r^{\pm})$ seems to be known.

(v) The case of an arbitrary symmetrizable Kac-Moody is much harder. Indeed, it is reasonable to expect that, if it exists,

$$\Delta : Y_\hbar \mathfrak{g} \longrightarrow Y_\hbar(\mathfrak{g}) \widehat{\otimes} Y_\hbar(\mathfrak{g})$$

(suitable completion)

* affine Yangians [GNW18]
* Maulik-Okounkov Yangians ($\equiv$ symmetric case)

(2.7) Drinfeld's classification of irreducible finite-dimensional representations of $Y_\hbar(\mathfrak{g})$

$\mathfrak{g}$ = f.d. simple Lie algebra, $Y := Y_\hbar(\mathfrak{g})$

<u>Def</u> A representation V of Y is a highest weight representation (with highest weight $\underline{\lambda} \in \mathbb{C}^{I \times \mathbb{N}}$)

$\underline{\lambda} = \{\lambda_{i,r}\}_{i\in I, r\in\mathbb{N}}$

if $\exists\, v \in V$ s.t.

(i) $x^+_{i,r} \cdot v = 0$

(ii) $\xi_{i,r} \cdot v = \lambda_{i,r} v$

(iii) $V = Y \cdot v$

<u>Example</u> Given $\underline{\lambda} = \{\lambda_{i,r} \in \mathbb{C}\}_{i\in I, r\in\mathbb{N}}$, the Verma module

$$M(\underline{\lambda}) := Y \big/ \left(x^+_{i,r},\ \xi_{i,r} - \lambda_{i,r} \;\middle|\; i\in I,\ r\in\mathbb{N} \right)$$

has a unique maximal proper submodule $M'(\underline{\lambda})$ and we set $L(\underline{\lambda}) := M(\underline{\lambda}) \big/ M'(\underline{\lambda})$.

Thm (1) Every irreducible f.d. representation of Y is h.w. (hence isomorphic to $L(\underline{\lambda})$ for some $\underline{\lambda}$).

(2) $L(\underline{\lambda})$ is f.d. if and only if there exist monic polynomials $P_i(u) \in \mathbb{C}[u]$ s.t.

$$\lambda(u) := 1 + \hbar \sum_{r \geq 0} \lambda_{i,r} \cdot u^{-r-1} = \frac{P_i(u + d_i \hbar)}{P_i(u)}$$

(expansion at $u = \infty$)

Example $P(u) = u - a \quad (a \in \mathbb{C})$

$$\frac{P(u+\hbar)}{P(u)} = 1 + \hbar \sum_{r \geq 0} a^r \cdot u^{-r-1}$$

pf (1) Let V be a f.d. irreducible representation of Y.

Then $Y \hookleftarrow \mathfrak{g} \supset \mathfrak{h}$ and as $\mathfrak{h}$-module

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu \longleftarrow \text{weight space}$$

Let $\lambda \in \mathfrak{h}^*$ be a maximal weight of V (i.e. $V_\lambda \neq 0$ and $V_\mu = 0$ for any $\mu > \lambda$).

By (Y2) $[\xi_{i,0}, x^+_{j,r}] = a_{ij}\, x^+_{i,r}$, we have

$$x^+_{i,r} \cdot V_\lambda = 0 \qquad \forall i \in I,\ r \in \mathbb{N}$$

By (Y1) $[\xi_{i,r}, \xi_{j,s}] = 0$, thus $\{\xi_{i,r} \mid i \in I, r \in \mathbb{N}\}$ is a commutative family of operators on the f.d. space $V_\lambda \implies \exists$ joint eigenvector $v \in V_\lambda$, i.e.

$$\xi_{i,r} \cdot v = \lambda_{i,r} \cdot v \quad \text{for some } \lambda_{i,r} \in \mathbb{C}$$

Finally, by irreducibility, $Y \cdot v = V$.

(2) The proof is by

- reduction to rank one case by restriction to $Y_{d_i\hbar}(\mathfrak{sl}_2) \hookrightarrow Y_\hbar(\mathfrak{g})$
- classification for $Y_\hbar(\mathfrak{sl}_2)$

$\square$

(2.8) Evaluation representations for $Y_\hbar(\mathfrak{sl}_2)$.

Set $Y := Y_\hbar(\mathfrak{sl}_2)$.

(2.8.1) Evaluation homomorphism.

For each $a \in \mathbb{C}$, we have an algebra map

$$ev_a : Y \longrightarrow U\mathfrak{sl}_2$$

s.t.

$$ev_a(\xi_0) := h \qquad ev_a(x_0^+) := e \qquad ev_a(x_0^-) := f$$

$$ev_a(t_1) := ah - \frac{\hbar}{2}(ef + fe)$$

Then, for any $\lambda \in \mathbb{C}$, $a \in \mathbb{C}$, we set

$M_\lambda(a) := ev_a^*(M_\lambda)$ ⇜ Verma over $\mathfrak{sl}_2$

$L_\lambda(a) := ev_a^*(L_\lambda)$ ⇜ irreducible over $\mathfrak{sl}_2$

$V(a) := ev_a^*(V)$ ⇜ $V \in \mathrm{Rep}(U\mathfrak{sl}_2)$

(2.8.2) Explicit description of $L_\lambda(a)$

Recall that M_λ has a basis $\{m_\lambda(r)\}_{r\geq 0}$ with $\mathfrak{sl}_2$-action given by

$$h\cdot m_\lambda(r) = (\lambda-2r)\,m_\lambda(r)$$

$$e\cdot m_\lambda(r) = (\lambda-r+1)\,m_\lambda(r-1) \qquad m_\lambda(-1) := 0$$

$$f\cdot m_\lambda(r) = (r+1)\,m_\lambda(r+1)$$

If $\lambda \notin \mathbb{N}$, M_λ is irreducible and $L_\lambda = M_\lambda$ is ∞-dim'l.

If $\lambda\in\mathbb{N}$, $e\cdot m_\lambda(\lambda+1)=0$, $M'_\lambda := (U\mathfrak{sl}_2)\cdot m_\lambda(\lambda+1)$ is a proper submodule and

$$L_\lambda := M_\lambda / M'_\lambda \simeq \operatorname{span}\{m_\lambda(r) \mid 0\leq r\leq \lambda\}$$

is the $(\lambda+1)$-dimensional irreducible representation.

Lemma Set $a_r := a + \hbar/2\,(\lambda-2r+1)$.

The action of Y on $M_\lambda(a)$ (and $L_\lambda(a)$) is given by

$$\xi(u)\cdot m_\lambda(r) = \frac{(u-a_0)(u-a_{\lambda+1})}{(u-a_r)(u-a_{r+1})}\cdot m_\lambda(r)$$

$$x^+(u)\cdot m_\lambda(r) = \hbar\,\frac{\lambda - r + 1}{u - a_r}\cdot m_\lambda(r-1)$$

$$x^-(u)\cdot m_\lambda(r) = \hbar\,\frac{r+1}{u - a_{r+1}}\cdot m_\lambda(r+1)$$

pf By $ev_a(t_1) := ah - \frac{\hbar}{2}(ef + fe)$, we get

$$t_1\cdot m_\lambda(r) = \left(a(\lambda - 2r) - \frac{\hbar}{2}\lambda - \hbar r(\lambda - r)\right)\cdot m_\lambda(r)$$

thus

$$ad(t_1) \circlearrowleft \hom\left(\mathbb{C}\, m_\lambda(r), \mathbb{C}\, m_\lambda(r-1)\right)$$

by $2a + \hbar(\lambda - 2r + 1) = 2a_r$

$$-a(\lambda - 2r) + \frac{\hbar}{2}\lambda + \hbar r(\lambda - r)$$

$$+ a(\lambda - 2r + 2) - \frac{\hbar}{2} - \hbar(r-1)(\lambda - r + 1)$$

Thus, from $\left(u \mp \frac{1}{2}\,ad(t_1)\right)\left(x^\pm(u)\right) = \hbar x_0^\pm$, we get

$$x^+(u)\cdot m_\lambda(r) = (u - a_r)^{-1}\hbar x_0^+\cdot m_\lambda(r)$$

$$= \hbar\,\frac{\lambda - r + 1}{u - a_r}\cdot m_\lambda(r-1)$$

In the same way, we get

$$x^-(u)\cdot m_\lambda(r) = \hbar\,\frac{r+1}{u-a_{r+1}}\cdot m_\lambda(r+1)$$

and finally $\xi(u) = 1 + [x^+(u), x_0^-]$. $\square$

(2.8.3) <u>Drinfeld polynomial for $L_\lambda(a)$</u> $(\lambda \in \mathbb{N})$

Set $b = a + \frac{\hbar}{2}(\lambda - 1)$. Then we get

$$a_r = a + \frac{\hbar}{2}(\lambda - 1 - 2r + 2) = b - \hbar(r-1)$$

$$\xi(u)\cdot m_\lambda(0) = \frac{u - b + \lambda\hbar}{u-b}\cdot m_\lambda(0)$$

$$\xi(u)\cdot m_\lambda(r) = \frac{(u-b-\hbar)(u-b+\lambda\hbar)}{(u-b+(r-1)\hbar)(u-b-r\hbar)}\cdot m_\lambda(r)$$

$$x^+(u)\cdot m_\lambda(r) = \hbar\,\frac{\lambda-r+1}{u-b+(r-1)\hbar}\cdot m_\lambda(r-1)$$

$$x^-(u)\cdot m_\lambda(r) = \hbar\,\frac{r+1}{u-b+r\hbar}\cdot m_\lambda(r+1)$$

Set $V(\lambda, b) := ev^*_{b - \hbar/2(\lambda-1)} L_\lambda$. Then

$$\sigma\big(V(\lambda,b)\big) = \begin{cases} \{b, b-\hbar, \dots, b-(\lambda-1)\hbar\} & \text{if } \lambda \in \mathbb{N} \\ \{b - r\hbar \mid r \geqslant 0\} & \text{if } \lambda \notin \mathbb{N} \end{cases}$$

$\uparrow$

poles of $\xi(u), x^{\pm}(u)$ on $V(\lambda, b)$

Finally note that if $\lambda \in \mathbb{N}$ the Drinfeld polynomial of $V(\lambda, b)$ is given by

$$P(u) = \prod_{k=0}^{\lambda-1} (u - b + k\hbar) =: P_{\lambda,b}(u)$$

since

$$\frac{P(u+\hbar)}{P(u)} = \frac{u - b + \lambda\hbar}{u-b}$$

Moreover, $\mathfrak{z}(P_{\lambda,b}) = \sigma(V(\lambda,b))$.

$\uparrow$

zeros of $P_{\lambda,b}$

References

Chari, Pressley, 1995

A guide to quantum groups (ch. 12)

Drinfeld, 1988

A new realization of Yangians and quantum loop algebras

Levendorskii, 1993

On generators and defining relations of Yangians

Guay, Nakajima, Wendlandt, 2018

Coproduct for Yangians of affine Kac-Moody algebras

Gautam, Toledano Laredo, Wendlandt, 2019

The meromorphic R-matrix of the Yangian