Lecture 3

(3.0) Last time

(3.0.1) We introduced the Drinfeld's presentation of $Y = Y_\hbar \mathfrak{sl}_2$ in terms of

$$x^\pm(u) := \hbar \sum_{r \geq 0} x_r^\pm u^{-r-1}$$

$$\xi(u) := 1 + \hbar \sum_{r \geq 0} \xi_r u^{-r-1}$$

(3.0.2) We reduced it to Levendorskii's presentation proving that Y is generated by $x_0^\pm$, $t_0 := \xi_0$, and $t_1 := \xi_1 - \hbar/2\, \xi_0^2$ with the recursive relations

$$x^\pm(u) = \left(u \mp \frac{1}{2} \operatorname{ad}(t_1)\right)^{-1} \hbar x_0^\pm$$

$$\xi(u) = 1 + [x^+(u), x_0^-]$$

(3.0.3) We used Levendorskii's presentation to describe the Hopf algebra structure of Y with coproduct $\Delta : Y \to Y \otimes Y$

$$\Delta(\xi_0) = \xi_0 \otimes 1 + 1 \otimes \xi_0$$
$$\Delta(x_0^{\pm}) = x_0^{\pm} \otimes 1 + 1 \otimes x_0^{\pm}$$
$$\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1 - 2\hbar\, x_0^{-} \otimes x_0^{+}$$

The counit and the antipode are as follows. We have an algebra map $\varepsilon : Y \to \mathbb{C}$ given by $\varepsilon(y_r) = 0$ with $y = \xi, x^{\pm}$ and $r \geq 0$, and a (co)algebra antiautomorphism $S : Y \to Y$ given by

$$S(y_0) = -y_0 \qquad y = \xi, x^{\pm}$$
$$S(t_1) = -t_1 - 2\hbar\, x_0^{-} x_0^{+}$$

(3.0.3) We introduced the evaluation map

$$ev_a : Y \longrightarrow U\mathfrak{sl}_2 \quad (a \in \mathbb{C})$$ given by

$$ev_a(\xi_0) := h \qquad ev_a(x_0^+) := e \qquad ev_a(x_0^-) := f$$

$$ev_a(t_1) := ah - \frac{\hbar}{2}(ef + fe)$$

and the evaluation representations

$$V(\lambda, b) := ev^*_{b - \hbar/2(\lambda - 1)} L_\lambda \qquad (\lambda, b \in \mathbb{C})$$

with basis $m_\lambda(r)$ $(0 \leq r < r^{top})$

$$r^{top} = \begin{cases} \infty & \text{if } \lambda \notin \mathbb{Z}_{\geq 0} \\ \lambda & \text{if } \lambda \in \mathbb{Z}_{\geq 0} \end{cases}$$

and action given by

$$\xi(u) \cdot m_\lambda(r) = \frac{(u - b - \hbar)(u - b + \lambda\hbar)}{(u - b + (r-1)\hbar)(u - b - r\hbar)} \cdot m_\lambda(r)$$

$$x^+(u) \cdot m_\lambda(r) = \hbar \frac{\lambda - r + 1}{u - b + (r-1)\hbar} \cdot m_\lambda(r-1)$$

$$x^-(u) \cdot m_\lambda(r) = \hbar \frac{r+1}{u - b + r\hbar} \cdot m_\lambda(r+1)$$

(3.0.4) $V(\lambda, b)$ is irreducible and it is finite-dimensional iff $\lambda \in \mathbb{Z}_{\geq 0}$. In this case, $\dim(V(\lambda,b)) = \lambda+1$ and

$$\xi(u) \cdot m_\lambda(0) = \frac{u-b+\lambda\hbar}{u-b} \cdot m_\lambda(0)$$

$$= \frac{P(u+\hbar)}{P(u)} \cdot m_\lambda(0)$$

$$P(u) = \prod_{k=0}^{\lambda-1} (u-b+k\hbar) =: P_{\lambda,b}(u)$$

(Drinfeld's polynomial)

Note that we have

$$\frac{u-b+\lambda\hbar}{u-b} = 1 + \lambda\hbar \sum_{r\geq 0} b^r u^{-r-1}$$

eigenvalues of ξ_r on $m_\lambda(0)$

and $x^+(u) \cdot m_\lambda(0) = 0$.

Remark

$V(0,b) \simeq \mathbb{C} \rightsquigarrow P_{0,b}(u) = 1$

$V(1,b) \simeq \mathbb{C}^2 \rightsquigarrow P_{1,b}(u) = u-b$ (fundamental)

$$\| $$

$ev_b^* \mathbb{C}^2 =: \mathbb{C}_b^2$ (note $\lambda=1$ so $b - \frac{\hbar}{2}(\lambda-1) = b$)

(3.0.5) Given $\underline{\lambda} = \{ \lambda_r \in \mathbb{C} \mid r \geq 0 \}$, we defined the Verma module

$$M(\underline{\lambda}) := \mathcal{Y} \big/ \left(x_r^+ , \xi_r - \lambda_r \mid r \in \mathbb{N} \right)$$

and its irreducible quotient

$$L(\underline{\lambda}) := M(\underline{\lambda}) \big/ M'(\underline{\lambda})$$

<u>Today</u> we complete the proof of the classification theorem for $\mathfrak{sl}_2$

<u>Thm</u> (1) Every irreducible f.d. representation of $\mathcal{Y}$ is h.w. (hence isomorphic to $L(\underline{\lambda})$ for some $\underline{\lambda}$). ✓

(2) $L(\underline{\lambda})$ is f.d. if and only if there exist a (unique) monic polynomial $P(u) \in \mathbb{C}[u]$ s.t.

$$\lambda(u) := 1 + \hbar \sum_{r \geq 0} \lambda_r \cdot u^{-r-1} = \frac{P(u+\hbar)}{P(u)}$$

(3.1) Tensor product of representations (with Drinfeld polynomials)

For the proof, we shall need few results about the tensor product of representations which have a Drinfeld polynomials (e.g. $V(\lambda, b)$).

Thus we need to understand the coproduct a bit more.

Lemma Set $wt(x_r^{\pm}) = \pm 2$ and $wt(\xi_r) = 0$.

Then,

$$\Delta(x^+(u)) = x^+(u) \otimes 1 + \xi(u) \otimes x^+(u)$$

$+$ elements of weights $(-2k) \otimes (2k+2)$ $\quad k \geq 1$

$$\Delta(\xi(u)) = \xi(u) \otimes \xi(u)$$

$+$ elements of weights

$(-2k) \otimes 2k$ $\quad k \geq 1$

$+$ $-2 \otimes 2k$

pf We need to prove that

$$\Delta(x_r^+) = x_r^+ \otimes 1 + 1 \otimes x_r^+ + \hbar \sum_{k=0}^{r-1} \xi_k \otimes x^+_{r-k-1} + \dots$$

We do it by induction on r by $2x^+_{r+1} = [t_1, x_r^\pm]$

and $\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1 - 2\hbar\, x_0^- \otimes x_0^+$.

Then one shows that

$$\Delta(\xi_r) = \xi_r \otimes 1 + 1 \otimes \xi_r + \hbar \sum_{k=0}^{r-1} \xi_k \otimes \xi_{r-k-1} + \dots$$

using $\xi_r = [x^+_r, x_0^-]$.

$\square$

Prop Let $V, W \in \mathrm{Rep}^{fd,hw}(Y)$ with Drinfeld polynomials $P_V, P_W \in \mathbb{C}[u]$. Then

$$\begin{array}{lll}
V & W & V \otimes W \in \mathrm{Rep}^{fd,hw}(Y) \\
|\uparrow\rangle & |\uparrow\rangle & |\uparrow\rangle_{V\otimes W} := |\uparrow\rangle \otimes |\uparrow\rangle =: |\uparrow\uparrow\rangle \\
P_V & P_W & P_{V\otimes W} := P_V \cdot P_W
\end{array}$$

Pf From the Lemma,

$$x^+(u) \cdot |\uparrow\uparrow\rangle = \Delta(x^+(u)) \cdot |\uparrow\rangle \otimes |\uparrow\rangle$$

$$= \left\{ x^+(u) \otimes 1 + \xi(u) \otimes x^+(u) + y \otimes y_{>0} \right\} \cdot |\uparrow\rangle \otimes |\uparrow\rangle = 0$$

and

$$\xi(u) \cdot |\uparrow\uparrow\rangle = \Delta(\xi(u)) \cdot |\uparrow\rangle \otimes |\uparrow\rangle$$

$$= \left\{ \xi(u) \otimes \xi(u) + y \otimes y_{>0} \right\} \cdot |\uparrow\rangle \otimes |\uparrow\rangle$$

$$= \frac{P_V(u+\hbar)\, P_W(u+\hbar)}{P_V(u)\, P_W(u)} |\uparrow\uparrow\rangle$$

$\square$

Remark From the classification theorem it follows that if V, W and $V \otimes W$ are f.d. irreducible representations, then

$$V \otimes W \simeq W \otimes V$$

However this won't be true in general.

(3.2) Proof of (3.0.5): $\exists P(u) \Rightarrow \dim L(\underline{\lambda}) < \infty$

Assume the irreducible representation $L(\underline{\lambda})$

admits $P(u) \in \mathbb{C}[u]$ monic s.t.

$$1 + \hbar \sum_{r \geq 0} \lambda_r u^{-r-1} = \frac{P(u+\hbar)}{P(u)}$$

and write $P(u) = \prod_{k=0}^{N} (u - a_k)$.

Set

$$\mathbb{C}^2_a := ev_a^* \mathbb{C}^2$$

$$V := \mathbb{C}^2_{a_1} \otimes \cdots \otimes \mathbb{C}^2_{a_N}$$

and let $|\uparrow \cdots \uparrow\rangle \in V$ be the tensor product of the h.w. vectors. Then by Lemma (3.1)

$$x^+(u) \cdot |\uparrow \cdots \uparrow\rangle = 0$$

$$\xi(u) \cdot |\uparrow \cdots \uparrow\rangle = \prod_{i=1}^{N} \frac{u - a_i - \hbar}{u - a_i}$$

Let $V' \subseteq V$ be the submodule generated by $|\uparrow \cdots \uparrow\rangle$. By universal property of $M(\underline{\lambda})$ we get a surjective map

$$\begin{tikzcd} M(\underline{\lambda}) \arrow[r, two heads, "\varphi"] \arrow[d, two heads] & V' \arrow[d, two heads] \\ L(\underline{\lambda}) \arrow[r, "\sim"] & V' / \varphi(M'(\underline{\lambda})) \end{tikzcd}$$

f.d. by constr.

unique mxl proper submodule

Remark As a corollary of the classification theorem we get that every f.d. irreducible representation of $\mathfrak{g}$ appears as a subquotient of a tensor product of fundamental representations.

(3.2) Proof of (3.0.5): $\dim L(\underline{\lambda}) < \infty \Rightarrow \exists P(u)$

This follows from a result due to A. Molev.

(3.2.1) Let $\mathcal{L}$ be an irreducible h.w. rep. of Y s.t.

$$\xi(u) \cdot |\uparrow\rangle = \frac{P_1(u)}{P_2(u)} |\uparrow\rangle$$

with $\deg(P_1) = \deg(P_2)$. List their zeros as

$$\text{zeros}(P_1) = \{ a_i \mid 1 \le i \le N \}$$

$$\text{zeros}(P_2) = \{ b_i \mid 1 \le i \le N \}$$

in such a way that the following holds

for any $1 \le k \le N$, if

$$(*) \qquad I_k := \{ b_k - a_j \mid 1 \le j \le k \} \cap \hbar \mathbb{Z}_{\geq 0} \neq \phi$$

then $b_k - a_k = \min I_k$

not necessarily in $\mathbb{Z}_{\geq 0}$

Thm. Set $\lambda_i := \dfrac{b_i - a_i}{\hbar} \in \mathbb{C} \quad (1 \le i \le N)$.

Then,

$$\mathcal{L} \simeq V(\lambda_1, b_1) \otimes \cdots \otimes V(\lambda_N, b_N)$$

(3.2.2) Let's finish the proof of the classification theorem relying on this result.

Let $L(\underline{\lambda}) \in Rep(Y)$ be a f.d. irreducible.

By finite-dimensionality, the action of $\xi(u)$ is rational, thus

$$\xi(u) \cdot |\uparrow\rangle = \frac{P_1(u)}{P_2(u)} |\uparrow\rangle$$

Note that

$$\lim_{u \to \infty} \frac{P_1(u)}{P_2(u)} = 1 = \xi(\infty)$$

and $\deg(P_1) = \deg(P_2)$. Enumerating the zeros as in (3.2.1) we get

$$L(\underline{\lambda}) \simeq V(\lambda_1, b_1) \otimes \cdots \otimes V(\lambda_N, b_N)$$

Since $\dim L(\underline{\lambda}) < \infty$, necessarily we get

$$\lambda_i \in \mathbb{Z}_{\geq 0} \implies a_i = b_i + \hbar \lambda_i$$

and every $V(\lambda_i, b_i)$ has a Drinfeld polynomial.

Thus, set

$$P(u) := \prod_{i=1}^{N} (u-b_i)\cdots(u-b_i+(\lambda_i-1)\hbar)$$

$$= \prod_{i=1}^{N} \underbrace{\prod_{k=0}^{\lambda_i-1} (u-b_i+k\hbar)}_{P_{\lambda_i,b_i}(u)}$$

$a_i = b_i + \lambda_i \hbar$

$$\frac{P_1(u)}{P_2(u)} = \prod_{i=1}^{N} \frac{u-b_i+\lambda_i\hbar}{u-b_i} = \frac{P(u+\hbar)}{P(u)}$$

Finally, note that

$$\frac{P(u+\hbar)}{P(u)} = \frac{Q(u+\hbar)}{Q(u)} \Longrightarrow \frac{Q(u)}{P(u)} = \frac{Q(u+\hbar)}{P(u+\hbar)}$$

$$\Longrightarrow Q(u) = P(u)$$

$\square$

(3.2.3) <u>Proof of (3.2.1)</u>

Set $V := V(\lambda_1, b_1) \otimes \cdots \otimes V(\lambda_N, b_N)$.

The cyclic span $W := \mathcal{Y} \cdot m_{\lambda_1}(0) \otimes \cdots \otimes m_{\lambda_N}(0)$ is h.w. with h.w. $P_1(u)/P_2(u)$. Thus, if we prove that V is irreducible we're done.

<u>Claim</u> If $\eta \in V$ s.t. $x^+(u) \cdot \eta = 0$, η is a scalar multiple of $m_{\lambda_1}(0) \otimes \cdots \otimes m_{\lambda_N}(0)$. Hence the only non-trivial submodule is W.

pf Write $\eta = \sum_{p=0}^{M} \eta_p \otimes m_{\lambda_N}(p)$.

Since $x_0^+ \cdot \eta = 0$,

$$\sum_{p=0}^{M} \Big(x_0^+ \eta_p \otimes m_{\lambda_N}(p) + \\ + (\lambda_N - p + 1) \eta_p \otimes m_{\lambda_N}(p-1) \Big) = 0$$

(1) Coeff. $m_{\lambda_N}(M)$: $\quad x_0^+ \cdot \eta_M = 0$

(2) Coeff. $m_{\lambda_N}(M-1)$: $\quad x_0^+ \eta_{M-1} = -(\lambda_N - M+1)\eta_M$

In the same way, one gets $x_r^+ \cdot \eta_M = 0$.

By induction on N (number of tensor factor)

we get $\eta_M = m_{\lambda_1}(0) \otimes \cdots \otimes m_{\lambda_{N-1}}(0)$ (up to a scalar).

Suppose $M > 0$. Then we get through $\Delta(x^+(u))$

(3) $\quad x^+(u)\cdot \eta_{M-1} + \xi(u)\cdot \eta_M \hbar \dfrac{\lambda_N - M+1}{u - b_N + (M-1)\hbar} = 0$

Set $\zeta_i := m_{\lambda_1}(0) \otimes \cdots \otimes m_{\lambda_i}(1) \otimes \cdots \otimes m_{\lambda_{N-1}}(0)$.

Then $\eta_{M-1} \in \operatorname{span}\{\zeta_i \mid 1 \leq i \leq N-1\}$ by (2).

We have

$$x^+(u)\cdot \zeta_i = \left(\prod_{j=1}^{i-1} \frac{u - b_j + \lambda_j \hbar}{u - b_j}\right) \frac{\hbar}{u - b_i}\, \eta_M$$

$$\xi(u)\cdot \eta_M = \prod_{i=1}^{N-1} \frac{u - a_i}{u - b_i} \cdot \eta_M$$

From (3) we must have

$$\prod_{i=1}^{N-1} \left(b_N - (M-1)\hbar - a_i \right) = 0$$

therefore $\exists\, j$ s.t. $b_N - a_j = (M-1)\hbar$ and by constuction $(*)$ it follows that

- $b_N - a_N = \lambda_N \hbar \in \mathbb{Z}_{\geq 0}\hbar$
- $\lambda_N \leq M-1$

but $\lambda_N \in \mathbb{Z}_{\geq 0}$ implies $\dim V(\lambda_N, b_N) = \lambda_N + 1$ and $M \leq \lambda_N$ $\lightning$

<u>Claim</u> $W = V$.

Pf Note that

$$V^* \simeq V(\lambda_1, b_1 - \hbar) \otimes \cdots \otimes V(\lambda_N, b_N - \hbar)$$

and satisfy $(*)$.

Indeed, the antipode gives an identification

$$M_\lambda(a)^* \simeq M_\lambda^*(a-\hbar)$$

we then identify the $\mathfrak{sl}_2$-modules via the Shapovalov form $M_\lambda^* \simeq M_\lambda$.

If $W \subsetneq V$, then in V^*

$$\mathrm{Ann}(W) := \{ \varphi \in V^* \mid \varphi(W) = 0 \}$$

is a proper non-trivial submodule, which does not contain the h.w. vector, thus contradicting claim 1 for V^* $\lightning$

References:

- Molev (ch. 3)
- Chari - Pressley (ch. 12)