Lecture 4

Last time

Thm (1) Every irreducible f.d. representation of $Y = Y_\hbar \mathfrak{sl}_2$ is h.w.
(hence isomorphic to $L(\underline{\lambda})$ for some $\underline{\lambda}$).

(2) $L(\underline{\lambda})$ is f.d. if and only if there exist a unique monic polynomial $P(u) \in \mathbb{C}[u]$ s.t.

$$\lambda(u) := 1 + \hbar \sum_{r \geq 0} \lambda_r \cdot u^{-r-1} = \frac{P(u+\hbar)}{P(u)}$$

(3) Every f.d. irreducible representation is isomorphic to a subquotient of a tensor product of fundamental reps $\mathbb{C}^2_{a_1} \otimes \cdots \otimes \mathbb{C}^2_{a_N}$

(4) Every f.d. irreducible representation is isomorphic to a tensor product of evaluation representations $V(\lambda_1, b_1) \otimes \cdots \otimes V(\lambda_N, b_N)$

(4.1) Irreducibility and strings of zeros

$V(\lambda, b)$ $(\lambda \in \mathbb{Z}_{\geq 0}, b \in \mathbb{C})$ is irreducible, finite-dimensional with $\dim(V(\lambda,b)) = \lambda+1$ and Drinfeld polynomial

$$P(u) = \prod_{k=0}^{\lambda-1} (u - b + k\hbar)$$

Thus, the zeros of $P(u)$ form a string of lenght λ (a sequence $z_0, z_1, \dots, z_{\lambda-1} \in \mathbb{C}$ s.t. $z_{k+1} - z_k = \hbar$)

$$S_{\lambda,b} := \{ z_k := b - k\hbar \in \mathbb{C} \mid k = 0, \dots, \lambda-1 \}$$

Def Two strings $S_1, S_2 \in \mathcal{S}$ are in special position if S_1, S_2 are not nested ($S_1 \cup S_2 \in \mathcal{S}$ and $S_1 \cup S_2 \neq S_1, S_2$). Otherwise, S_1, S_2 are in general position (or nested).

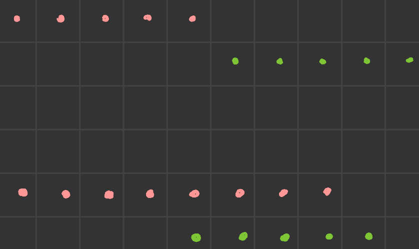
Example $S_{\lambda,b}$, $S_{\mu,c}$ are in special position iff $\exists\ 0 < k \leq \min(\lambda,\mu)$ s.t.

$$b - c = \hbar(\lambda - k + 1) \quad \text{or} \quad c - b = \hbar(\mu - k + 1)$$

special :

1)

2)



general :

1)

2)

Rmk $\forall$ multiset $S \subseteq \mathbb{C}$ $\exists!\ S_i \in \mathcal{S}$ s.t. S_i are pairwise in general position and $S = \bigcup S_i$

Thm (Chari - Pressley 90)

(1) $V(\lambda_1, b_1) \otimes \cdots \otimes V(\lambda_N, b_N)$ is irreducible iff $S_{\lambda_1,b_1}, \dots, S_{\lambda_N,b_N}$ are in general position

(2) Two irreducible tensor products $\bigotimes_i V(\lambda_i, b_i)$ and $\bigotimes_j V(\mu_j, c_j)$ are isomorphic iff they differ by a permutation of factors.

(4.2) The simplest example

Let's study the action of $\mathcal{Y}$ on $\mathbb{C}^2_{b_1} \otimes \mathbb{C}^2_{b_2}$

$$x^+(u) \qquad \begin{bmatrix} \dfrac{\hbar}{u-b_1} \\[2ex] \dfrac{\hbar(u-b_1-\hbar)}{(u-b_1)(u-b_2)} \end{bmatrix} \qquad \qquad \begin{bmatrix} \dfrac{\hbar(u-b_1+\hbar)}{(u-b_1)(u-b_2)}, & \dfrac{\hbar}{u-b_1} \end{bmatrix}$$

$$|\downarrow\downarrow\rangle \quad \rightleftarrows \quad \begin{matrix} |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \end{matrix} \quad \rightleftarrows \quad |\uparrow\uparrow\rangle$$

$$x^-(u) \qquad \begin{bmatrix} \dfrac{\hbar(u-b_2-\hbar)}{(u-b_1)(u-b_2)}, & \dfrac{\hbar}{u-b_2} \end{bmatrix} \qquad \qquad \begin{bmatrix} \dfrac{\hbar}{u-b_2} \\[2ex] \dfrac{\hbar(u-b_2+\hbar)}{(u-b_1)(u-b_2)} \end{bmatrix}$$

and finally $\xi(u)$

$$|\downarrow\downarrow\rangle \;\rightleftarrows\; \begin{matrix}|\uparrow\downarrow\rangle\\ |\downarrow\uparrow\rangle\end{matrix} \;\rightleftarrows\; |\uparrow\uparrow\rangle$$

$$\frac{(u-b_1-\hbar)(u-b_2-\hbar)}{(u-b_1)(u-b_2)} \qquad\qquad \frac{(u-b_1+\hbar)(u-b_2+\hbar)}{(u-b_1)(u-b_2)}$$

$$\begin{bmatrix} \dfrac{(u-b_1+\hbar)(u-b_2-\hbar)}{(u-b_1)(u-b_2)} & 0 \\[2ex] \dfrac{-2\hbar^2}{(u-b_1)(u-b_2)} & \dfrac{(u-b_1-\hbar)(u-b_2+\hbar)}{(u-b_1)(u-b_2)} \end{bmatrix}$$

$$\mathbb{C}^2_{a+\hbar} \otimes \mathbb{C}^2_a \qquad (b_1 = a+\hbar,\ b_2 = a)$$

$$\begin{tikzcd}[column sep=huge]
{|\downarrow\downarrow\rangle} \arrow[r, shift left, "{\begin{bmatrix} \frac{\hbar}{u-a-\hbar} \\ \frac{\hbar(u-a-2\hbar)}{(u-a-\hbar)(u-a)} \end{bmatrix}}"] & {|\uparrow\downarrow\rangle,\ |\downarrow\uparrow\rangle} \arrow[l, shift left, "{\begin{bmatrix} \frac{\hbar}{u-a} , \frac{\hbar}{u-a} \end{bmatrix}}"] \arrow[r, shift left, "{\begin{bmatrix} \frac{\hbar}{u-a-\hbar} , \frac{\hbar}{u-a-\hbar} \end{bmatrix}}"] & {|\uparrow\uparrow\rangle} \arrow[l, shift left, "{\begin{bmatrix} \frac{\hbar}{u-a} \\ \frac{\hbar(u-a+\hbar)}{(u-a-\hbar)(u-a)} \end{bmatrix}}"]
\end{tikzcd}$$

$$\begin{tikzcd}
{|\downarrow\downarrow\rangle} \arrow[loop below, "\frac{u-a-2\hbar}{u-a}"] \arrow[r, shift left] & {|\uparrow\downarrow\rangle,\ |\downarrow\uparrow\rangle} \arrow[l, shift left] \arrow[r, shift left] \arrow[loop below] & {|\uparrow\uparrow\rangle} \arrow[l, shift left] \arrow[loop below, "\frac{u-a+\hbar}{u-a-\hbar}"]
\end{tikzcd}$$

$$\begin{bmatrix} 1 & 0 \\ \frac{-2\hbar^2}{(u-a-\hbar)(u-a)} & \frac{(u-a-2\hbar)(u-a+\hbar)}{(u-a-\hbar)(u-a)} \end{bmatrix}$$

non-split

$$0 \to \mathbb{1} \to \mathbb{C}^2_{a+\hbar} \otimes \mathbb{C}^2_a \to V(2, a+\hbar) \to 0$$

trivial subrep. gen. by $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

$L_2(a+\hbar/2)$

$$\mathbb{C}^2_a \otimes \mathbb{C}^2_{a+\hbar} \qquad (b_1 = a,\ b_2 = a+\hbar)$$

$$\begin{tikzcd}[column sep=huge]
{|\downarrow\downarrow\rangle} \arrow[r, shift left, "{\begin{bmatrix} \frac{\hbar}{u-a} \\ \frac{\hbar}{(u-a)} \end{bmatrix}}"] & {|\uparrow\downarrow\rangle,\ |\downarrow\uparrow\rangle} \arrow[l, shift left, "{\begin{bmatrix} \frac{\hbar(u-a-2\hbar)}{(u-a)(u-a-\hbar)} , \frac{\hbar}{u-a-\hbar} \end{bmatrix}}"] \arrow[r, shift left, "{\begin{bmatrix} \frac{\hbar(u-a+\hbar)}{(u-a)(u-a-\hbar)} , \frac{\hbar}{u-a} \end{bmatrix}}"] & {|\uparrow\uparrow\rangle} \arrow[l, shift left, "{\begin{bmatrix} \frac{\hbar}{u-a-\hbar} \\ \frac{\hbar}{u-a-\hbar} \end{bmatrix}}"]
\end{tikzcd}$$

$$\begin{tikzcd}
{|\downarrow\downarrow\rangle} \arrow[loop below, "\frac{u-a-2\hbar}{u-a-\hbar}"] \arrow[r, shift left] & {|\uparrow\downarrow\rangle,\ |\downarrow\uparrow\rangle} \arrow[l, shift left] \arrow[r, shift left] \arrow[loop below] & {|\uparrow\uparrow\rangle} \arrow[l, shift left] \arrow[loop below, "\frac{u-a+\hbar}{u-a-\hbar}"]
\end{tikzcd}$$

$$\begin{bmatrix} \frac{(u-a+\hbar)(u-a-2\hbar)}{(u-a)(u-a-\hbar)} & 0 \\ \frac{-2\hbar^2}{(u-a)(u-a-\hbar)} & 1 \end{bmatrix}$$

non-split

$$0 \to V(2, a+\hbar) \to \mathbb{C}^2_a \otimes \mathbb{C}^2_{a+\hbar} \to \mathbb{1} \to 0$$

3 dim'l subrep. with basis $\{|\uparrow\uparrow\rangle,\ |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle,\ |\downarrow\downarrow\rangle\}$

(4.3) Lack of semisimplicity and braiding

From the previous example we have that

1. $\mathrm{Rep}^{fd}(Y)$ is not semisimple
2. $\mathrm{Rep}^{fd}(Y)$ is not braided

More generally, we have the following

Prop (Chari-Pressley 90)

$V := V(\lambda, b) \otimes V(\mu, c)$ is reducible iff

$\exists\ 0 < k \leq \min(\lambda, \mu)$ s.t.

$$b - c = \hbar(\lambda - k + 1) \quad \text{or} \quad c - b = \hbar(\mu - k + 1)$$

In this case, $\exists!\ 0 \neq W \subsetneq V$ and

Case 1: $b - c = \hbar(\lambda - k + 1)$

$W \qquad V(\lambda - k, b + \hbar k) \otimes V(\mu - k, c - \hbar k)$

$$V/W \qquad V(k-1, b - \hbar(\lambda - k + 1)) \otimes \\ \otimes V(\lambda + \mu - k + 1, c + \hbar(\lambda - k + 1))$$

Case 2 : $c - b = \hbar(\mu - k + 1)$

$$W \qquad V(k-1, b + \hbar(\lambda - k + 1)) \otimes \otimes V(\lambda + \mu - k + 1, c - \hbar(\lambda - k + 1))$$

$$V/W \qquad V(\lambda - k, b - \hbar k) \otimes V(\mu - k, c + \hbar k)$$

In particular, if $S_{\lambda,b}$ and $S_{\mu,c}$ are in special position, then

- $V(\lambda, b) \otimes V(\mu, c)$ is reducible
- $V(\lambda, b) \otimes V(\mu, c) \not\simeq V(\mu, c) \otimes V(\lambda, b)$

Remark (1) in our examples, $\lambda = 1 = \mu$

$$\mathbb{C}^2_{a+\hbar} \otimes \mathbb{C}^2_a \rightsquigarrow \text{case 1 with } k = 1$$

$$\mathbb{C}^2_a \otimes \mathbb{C}^2_{a+\hbar} \rightsquigarrow \text{case 2 with } k = 1$$

(2) The lack of semisimplicity is not a quantum phenomenon, since the same happens for $\mathrm{Rep}^{fd}(\mathfrak{g}[u])$ e.g.

$$J_k := \mathfrak{g}[u] \big/ (u-a)^{k+1} \cdot \mathfrak{g}[u] \supseteq (u-a) J_k$$

yet tensor products of evaluation representations of $\mathfrak{g}[u]$ are always semisimple.

$$\begin{array}{ccc} & & \mathcal{U}(\mathfrak{g}) \\ & \swarrow & \| \end{array}$$

(*) The evaluation map $ev : Y_\hbar \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ exists only for $\mathfrak{g} \simeq \mathfrak{sl}_n$.

In other types, one consider instead the fundamental representations $V(\lambda_i, b)$ with Drinfeld polynomials $P_j(u) = 1$ $(j \neq i)$ and $P_i(u) = u - b$.

(4.4) R-matrix

Recall that $\mathfrak{g}[u]$ is a Lie bialgebra with cobracket

f.d. semisimple

$$\delta : \mathfrak{g}[u] \longrightarrow \mathfrak{g}[u] \otimes \mathfrak{g}[u] = (\mathfrak{g} \otimes \mathfrak{g})[u,v]$$

given by

$$\delta(p) := \left[p(u) \otimes 1 + 1 \otimes p(v) , \frac{\Omega}{u-v} \right]$$

Casimir tensor of $\mathfrak{g}$

Remark $\mathfrak{g}[u] = \bigoplus_{r \geq 0} \mathfrak{g} u^r$ is $\mathbb{N}$-graded as a Lie algebra, while δ lowers the degree by 1. Drinfeld proved that $\exists!$ homogeneous quantization of $(\mathfrak{g}[u], \delta)$ $(\deg(\hbar) = 1)$ and this is precisely $Y_\hbar(\mathfrak{g})$.

Thm. (Drinfeld)

$\exists!\ R(u) \in Y \otimes Y [\![u^{-1}]\!]$ of the form

$$R(u) = 1 + \hbar \frac{\Omega}{u} + \mathcal{O}(u^{-2})$$

s.t.

(i) intertwining property : $\forall x \in Y$

$$\tau_u \otimes 1 \circ \Delta^{op}(x) = R(u) \left(\tau_u \otimes 1 \circ \Delta(x) \right) R(u)^{-1}$$

(ii) coupling identities :

$$\Delta \otimes 1 \left(R(u) \right) = R_{13}(u) R_{23}(u)$$

$$1 \otimes \Delta \left(R(u) \right) = R_{12}(u) R_{13}(u)$$

(iii) unitarity : $R(u)^{-1} = R_{21}(-u)$

(iv) shift invariance :

$$\tau_a \otimes \tau_b \left(R(u) \right) = R(u + a - b)$$

Remarks (1) Drinfeld's proof is cohomological in nature (thus non-explicit) and still unpublished (handwritten notes in russian from 1989 translated by P. Etingof).

(2) The convergence of $R(u)$ on a tensor product of f.d. representations is a hard problem!

Here an important difference appear between $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$. The quantum loop algebra is equipped with a family of multiplicative shift automorphisms σ_α and a spectral R-matrix $R(z) \in U_q \otimes U_q[\![z]\!]$ s.t.

$$\sigma_\alpha \otimes \sigma_\beta (R(z)) = R(z\alpha\beta^{-1})$$

Crossing symmetry for $\mathcal{U}_q$

For any $V \in \mathrm{Rep}^{fd}\,\mathcal{U}_q$,

$$V(z)^{**} \xrightarrow[\sim]{q^{-2\rho}} V(q^{2mh^\vee} \cdot z)$$

thus, since $S \otimes 1(R(z)) = R(z)^{-1}$, one gets a multiplicative q-difference equation

$$R_{VW}(q^{2mh^\vee} \cdot z) = \Phi(R_{VW}(z))$$

where for $f \circlearrowright V \otimes W$ ($f^{t_1} \circlearrowright V^* \otimes W$)

$$\Phi(f) = q^{-2\rho} \otimes 1 \cdot \left(\left(\left(f^{-1}\right)^{t_1}\right)^{-1}\right)^{t_1} \cdot q^{2\rho} \otimes 1$$

$\Longrightarrow$ formal solutions converge, thus $R_{VW}(z)$ is analytic near 0 and meromorphic on $\mathbb{C}$.

However, for $Y_\hbar(\mathfrak{g})$, the crossing symmetry is an additive difference equation and formal solutions do not necessarily converge!

(3) New constructive proof by Gautam, Toledano Laredo, and Wendlandt (2019) for $\mathfrak{g}$ semisimple and f.d. representations and by Maulik and Okounkov (2012) for $\mathfrak{g}$ symmetric Kac-Moody and representations of geometric origin.

poles of $x^+(u)$ and $\xi(u)$

(4.5) Drinfeld coproduct

Let $V_1, V_2 \in \mathrm{Rep}^{fd}\, \mathcal{Y}$ with $\sigma(V_1) \cap \sigma(V_2) = \phi$.

There is another action of $\mathcal{Y}$ on $V_1 \otimes V_2$ given by

$$\xi(u) \longrightarrow \xi(u) \otimes \xi(u)$$

$$x^+(u) \longrightarrow x^+(u) \otimes 1 + \oint_{C_2} \frac{1}{u-v}\, \xi(v) \otimes x^+(v)\, dv$$

$$x^-(u) \longrightarrow 1 \otimes x^-(u) +$$

$$+ \oint_{C_1} \frac{1}{u-v} \, x^-(v) \otimes \xi(v) \, dv$$

where C_i is a path which encloses $\sigma(V_i)$ and avoids $\sigma(V_j)$ $(i \neq j)$ and u.

The resulting representation is denoted $V_1 \underset{D}{\otimes} V_2$. In general, we get a family of representations

$$V_1(s) \underset{D}{\otimes} V_2$$

with poles at $s \in \sigma(V_2) - \sigma(V_1)$.

$$\shortparallel$$

$$\{ a_2 - a_1 \in \mathbb{C} \mid a_j \in \sigma(V_j) \}$$

(4.6) GTL / GTLW approach to $R(u)$

<u>Step 1</u> $\left(\mathrm{Rep}^{fd}\, Y, \underset{D}{\otimes}\right)$ is a meromorphic braided category

<u>Step 2</u> $\otimes$ and $\underset{D}{\otimes}$ are meromorphically twist equivalent

<u>Step 3</u> $V_1, V_2 \in \mathrm{Rep}^{fd}\, Y$

$$\begin{tikzcd}[column sep=huge, row sep=large]
V_1(s) \otimes V_2 \arrow[r, "(12)\circ R^{\uparrow/\downarrow}_{V_1V_2}(s)"] \arrow[d, "R^{-}_{V_1V_2}(s)"'] & V_2 \otimes V_1(s) \arrow[d, "R^{-}_{V_2V_1}(-s)"] \\
V_1(s) \underset{D}{\otimes} V_2 \arrow[r, "(12)\circ R^{0\uparrow/\downarrow}_{V_1V_2}(s)"'] & V_2 \underset{D}{\otimes} V_1(s)
\end{tikzcd}$$

<u>Step 4</u> for $s \to \infty$ in any half plane of the form $\pm \mathrm{Re}(s/\hbar) \gg 0$

$$R^{\uparrow/\downarrow}_{V_1V_2}(s) \sim R_{V_1V_2}(s)$$

Thm. (Gautam–Toledano Laredo 2015)

There exists meromorphic functions

$$\mathcal{R}^{0,\pm}_{V_1V_2} : \mathbb{C} \longrightarrow \mathrm{end}(V_1 \otimes V_2)$$

s.t.

(1) $\mathcal{R}^{0,\pm}_{V_1V_2}(s)$ is holomorphic and invertible for $\pm \mathrm{Re}(s/\hbar) \gg 0$, it's abelian

$$[\mathcal{R}^{0,\pm}_{V_1V_2}(s), \mathcal{R}^{0,\pm}_{V_1V_2}(s')] = 0$$

and

$$\mathcal{R}^{0,\pm}_{V_1V_2}(s) \sim 1 + \hbar/u\, \Omega_0 + O(s^{-2})$$

for $s \to \infty$ in $\pm \mathrm{Re}(s/\hbar) \gg 0$

(2) $(12) \circ \mathcal{R}^{0,\pm}_{V_1V_2}(s) : \underset{D}{V_1(s) \otimes V_2} \longrightarrow \underset{D}{V_2 \otimes V_1(s)}$

is a morphism in $\mathrm{Rep}\, \mathcal{Y}$

(3) $\mathcal{R}^{0,+}_{V_1V_2}(s)^{-1} = (12) \circ \mathcal{R}^{0,-}_{V_2V_1}(-s) \circ (12)$

(4) $V_1, V_2, V_3 \in \mathcal{R}ep^{fd}\, Y$

$$\mathcal{R}^{0,\varepsilon}_{V_1(s_1)\otimes V_2,\, V_3}(s_2) = \mathcal{R}^{0,\varepsilon}_{V_1V_3}(s_1+s_2)\,\mathcal{R}^{0,\varepsilon}_{V_2V_3}(s_2)$$

$$\mathcal{R}^{0,\varepsilon}_{V_1,\, V_2(s_2)\otimes V_3}(s_1+s_2) = \mathcal{R}^{0,\varepsilon}_{V_1V_3}(s_1+s_2)\,\mathcal{R}^{0,\varepsilon}_{V_1V_2}(s_1)$$

(5)

$$\mathcal{R}^{0,\varepsilon}_{V_1(a)V_2(b)}(s) = \mathcal{R}^{0,\varepsilon}_{V_1V_2}(s+a-b)$$

Sketch:

$$A(s) := \exp\left[-\oint_{C_1} \xi(u)^{-1}\xi'(u) \otimes \log \xi(u+s+\hbar)\, du\right]$$

$$\mathcal{R}^{0,\pm}(s+2\hbar) = A(s)\,\mathcal{R}^{0,\pm}(s)$$

$$\mathcal{R}^{0,+}(s) = \overrightarrow{\prod_{n\geq 1}} A(s+2n\hbar)^{-1}$$

$$\mathcal{R}^{0,-}(s) = \overrightarrow{\prod_{n\geq 1}} A(s-2n\hbar)$$

Thm (Gautam, Toledano Laredo, Wendlandt)

$\exists!\ \mathcal{R}^{-}(s) \in \mathcal{Y}^{-} \otimes \mathcal{Y}^{+} [\![s]\!]$ s.t.

$$(\mathrm{Rep}^{fd}\, \mathcal{Y}, \otimes) \sim (\mathrm{Rep}^{fd}\, \mathcal{Y}, \otimes_{D})$$

References

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"Meromorphic tensor equivalence for Yangians and quantum loop algebras" (2015)

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