Motivation

Let X be a smooth quasi-projective complex algebraic variety

Assume that $\exists$ action $\mathbb{C}^* \curvearrowright X$.

Let $X^{\mathbb{C}^*}$ be the fixed point locus. Let $i : X^{\mathbb{C}^*} \hookrightarrow X$ be the inclusion.

We will see that the theory of equivariant cohomology guarantees that $\exists$

$$i^* : H^*_{\mathbb{C}^*}(X) \longrightarrow H^*_{\mathbb{C}^*}(X^{\mathbb{C}^*})$$

Maulik-Okounkov's goal: construct a map in the other direction:

$$\mathrm{Stab} : H^*_{\mathbb{C}^*}(X^{\mathbb{C}^*}) \longrightarrow H^*_{\mathbb{C}^*}(X) \quad \text{- "stable envelope"}$$

Plan of the lectures:

1. The (co)homology theories we need to work with
2. Actions on varieties $\rightsquigarrow$ We need to move to the "equivariant world"
3. Symplectic geometry and relations with (1) and (2)
4. Stable envelopes: existence and unicity
5. —— " ——: Symplectic resolutions case

Borel-Moore homology

Goal: define a homology theory $H_*^{BM}(-)$, with complex coefficients, for complex algebraic/analytic varieties, such that:

- covariant w.r.t. proper morphisms: $f: X \longrightarrow Y$ proper $\Rightarrow H_*^{BM}(X) \xrightarrow{f_*} H_*^{BM}(Y)$
- contravariant w.r.t. l.c.i. morphisms: $f: X \longrightarrow Y$ l.c.i. $\Rightarrow H_*^{BM}(Y) \xrightarrow{f^*} H_*^{BM}(X)$
- $\exists$ fundamental class $[X] \in H_{2\dim(X)}^{BM}(X)$

Attention: differences with singular (co)homology theory

1. Given a continuous map $f: X \to Y$ between topological spaces:

$$H_*(-) \text{ is } \underline{\text{covariant}}, \text{ i.e., } H_*(X) \xrightarrow{f_*} H_*(Y)$$

$$H^*(-) \text{ is } \underline{\text{contravariant}}, \text{ i.e., } H^*(Y) \xrightarrow{f^*} H^*(X)$$

2. Let X be a projective smooth irreducible complex algebraic variety of dim n. Then X^{2n} is a compact oriented $2n$-dimensional real manifold. Thus, $\boxed{H_{2n}(X) \simeq \mathbb{Z}}$ with generator the fundamental class $[X]$.

References for Borel-Moore homology:

- [CG] Chriss, Ginzburg - Representation theory and complex geometry (Chapter 2)
- [F1] Fulton - Intersection Theory
- [F2] Fulton - Young Tableaux (Appendix B)

Basic Reference for Algebraic Geometry: [H] Hartshorne, Algebraic Geometry (Chapter 1)

Assumption: let X be a complex algebraic (/analytic) variety. From now on, simply "variety"

Attention

1. This assumption is very strong: one can work with a large class of "reasonable" spaces (see [CG, Section 2.6]).
2. It would be clear that for a complex algebraic variety X, $H_*^{BM}(X) = H_*^{BM}(X^{an})$ (X^{an} = analytification of X - see [H, Appendix B])

We are going to give 3 equivalent definitions of the Borel-Moore homology $H_*^{BM}(X)$ of a variety X:

- Def. "Locally Finite"
- — " — "1-point compactification"
- — " — "Poincaré duality"

Remark:

1. In [CG] you can find all the references, where the equivalences between these defs are proved.
2. $\exists$ a sheaf theoretic definition of the BM homology: $H_i^{BM}(X) = H^{-i}(X, \mathbb{D}_X)$
$\mathbb{D}_X$ = dualizing sheaf - cf. [CG, Section 8.3, Formula (8.3.7)]

Definition (LF = Locally finite chains)

Fix $i \in \mathbb{Z}_{\geq 0}$. Let $C_i^{BM}(X)$ be the free abelian group with basis the $\underbrace{\text{formal infinite}}_{\text{in contrast with singular homology}}$ sums

$$\sum_{\sigma} a_\sigma \sigma$$

where $a_\sigma \in \mathbb{C}$, $\sigma : \Delta^i \longrightarrow X$ continuous, and the sum is finite in the following sense:

$$\boxed{\forall D \subset X \text{ compact}, \exists \text{ only finitely many } a_\sigma \neq 0 \text{ s.t. } D \cap \sigma(\Delta^i) \neq \phi}$$

We have a chain complex: $\cdots \to C_2^{BM}(X) \xrightarrow{\partial} C_1^{BM}(X) \xrightarrow{\partial} C_0^{BM}(X) \longrightarrow 0$

where ∂ is the boundary map of singular chains.

Set $\boxed{H_i^{BM}(X) := Ker\left(\partial : C_i^{BM}(X) \longrightarrow C_{i-1}^{BM}(X)\right) / Im\left(\partial : C_{i+1}^{BM}(X) \longrightarrow C_i^{BM}(X)\right)}$

Exercise: If X is compact (as a topological space): $C_*^{BM}(X) = C_*(X) \implies H_i^{BM}(X) = H_i(X)$

Definition (1PC = 1-point compactification)

$\hat{X} = X \cup \{\infty\}$ = 1-point compactification of X. Then

$$\boxed{H_*^{BM}(X) \simeq H_*(\hat{X}, \infty)} \quad \left(= \text{singular relative homology of the pair } (\hat{X}, \infty)\right)$$

Definition (PD = Poincaré duality)

Let M be a smooth oriented manifold of $\dim_{\mathbb{R}} M = m$. Let $X \hookrightarrow M$ be a closed embedding as a topological space. Then

$$\boxed{H_i^{BM}(X) \simeq H^{m-i}(M, M \setminus X)}$$

cf. Aquistapace, Broglia, Tognoli, Annali della SNS 1979

Remark

1. $H_*^{BM}(X)$ does not depend on the embedding (see e.g. Fulton, Young Tableaux. Appendix B, Lemmas 1, 2)
2. If X is smooth, one may choose $M = X$. Thus we have

$$\boxed{H_i^{BM}(X) \simeq H^{m-i}(X)}$$

Functorial properties of Borel-Moore homology

▶ Proper pushforward.

Definition We say that a continuous map $f: X \to Y$ between topological spaces is *proper* if $\forall D \subset Y$ compact, $f^{-1}(D) \subset X$ is compact as well.

Let $f: X \to Y$ be a continuous map between locally compact Hausdorff spaces. f is proper $\Longleftrightarrow$ f is universally closed.

Remark $\exists$ a notion of proper morphism between algebraic varieties. Note that if $f: X \to Y$ is a proper morphism between algebraic varieties, $f: X^{an} \longrightarrow Y^{an}$ is proper in the topological sense (for the complex topology).

Example: A closed embedding between algebraic varieties is a proper morphism - [H, Chapter II, Corollary 4.8 - (a)).

Proposition Let X, Y be algebraic varieties. Let $f: X^{an} \to Y^{an}$ be a proper map. Then, there exists

proper pushforward map $\boxed{f_* : H_*^{BM}(X) \longrightarrow H_*^{BM}(Y)}$

Remark

1. Use definition 1PC to prove the existence.
2. $H_*^{BM}(-)$ is covariant w.r.t. proper maps, i.e., one has $(g \circ f)_* = g_* \circ f_*$ for $g: Y \to Z$ proper.

Long exact sequence.

We observe that for an open embedding $U \xrightarrow{J} X$ $\exists$ a natural restriction morphism:

$$\boxed{J^*: \; H_*^{BM}(X) \longrightarrow H_*^{BM}(U)}$$

Remark: let $Y := X \setminus U$ and $M^\circ = M \setminus Y$. Then j^* is given by

$$H^{m-i}(M, M \setminus X) \longrightarrow H^{m-i}(M^\circ, M^\circ \setminus U)$$

Consider a closed embedding $i: Y \hookrightarrow X$, set $U = X \setminus F$. Thus, we have

$$Y \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} F$$

There exists a natural long exact sequence:

$$\boxed{\cdots \longrightarrow H_p^{BM}(Y) \xrightarrow{i_*} H_p^{BM}(X) \xrightarrow{j^*} H_p^{BM}(U) \longrightarrow H_{p-1}^{BM}(Y) \longrightarrow \cdots}$$

Remark:

1. By using def PD one proves the existence of the long exact sequence.
2. Let $X = X_1 \sqcup \cdots \sqcup X_n$. Then

$$\boxed{H_i^{BM}(X) = H_i^{BM}(X_1) \oplus \cdots \oplus H_i^{BM}(X_n)}$$

▶ Fundamental class.

A fundamental feature of BM homology is the existence of the fundamental class $[X]$ for any (not necessarily smooth or compact) variety X.

Proposition. Let X be a complex algebraic variety of dimension n. Then

- $H_i^{BM}(X) = 0$ for $i > 2n$, and
- $H_{2n}^{BM}(X)$ is a free abelian group with a generator for each n-dimensional irreducible component of X.

Proof

all irreducible components are n-dimensional
$\parallel$
Assume first that X is smooth and purely n-dimensional. In this case X is an oriented $2n$ manifold. Thus

$$\begin{cases} H_i^{BM}(X) \simeq H^{2n-i}(X) = 0 & \text{for } i > 2n \\ H_{2n}^{BM}(X) \simeq H^0(X) \end{cases}$$

and the assertion follows by standard algebraic topology. Indeed, recall that for a smooth complex algebraic variety, its connected components are the same as its irreducible ones (standard exercise).

Now, we prove the general statement by induction on the dimension n.

Let Y be the union of all irreducible components of X of dimension less than n, together with the singular locus of X.

Fact: Y is of dimension less then n and $U = X \setminus Y$ is smooth and purely n-dimensional

Thus,

- $H_i^{BM}(Y) = 0$ for $i > 2n-2$, by induction hypothesis
- $H_i^{BM}(U) = 0$ for $i > 2n$, since U is smooth (thus we apply the first case we have discussed)

By the long exact sequence relative to $Y \hookrightarrow X \hookleftarrow U$, we get:

$$0 = H_{2n}^{BM}(Y) \longrightarrow H_{2n}^{BM}(X) \longrightarrow H_{2n}^{BM}(U) \longrightarrow H_{2n-1}^{BM}(Y) = 0$$

Thus $H_i^{BM}(X) = 0$ for $i > 2n$ and $H_{2n}^{BM}(X) \simeq H_{2n}^{BM}(U)$, which is freely generated by the irreducible components of U, which are exactly the restrictions of the n-dimensional components of X. $\square$

Remark:

1. $H_*^{BM}(-)$ is not homotopy invariant: e.g., $H_i^{BM}(\mathbb{C}^n) \simeq \begin{cases} \mathbb{C}[\mathbb{C}^n] & \text{for } i = 2n; \\ 0 & \text{otherwise.} \end{cases}$
2. Let X be an irreducible variety, let X^{sm} be the smooth locus of X. Then

$$\boxed{H_{2\dim(X)}^{BM}(X) \simeq H_{2\dim(X)}^{BM}(X^{sm})}$$

Let $X_1, \dots, X_r$ be the n-dimensional irreducible components of X. Denote: $\boxed{[X] := \sum_{i=1}^{r} [X_i]}$

<u>Definition:</u> Let $Y \subset X$ be a closed subvariety. We call the fundamental class of Y in X the image of $[Y]$ via the proper pushforward $H^{BM}_{2\dim(Y)}(Y) \longrightarrow H^{BM}_{2\dim(Y)}(X)$.

▶ Pairing

<u>Proposition</u> Sia X be a smooth irreducible (not necessarily compact) variety of dimension n. Then there exists a non-degenerate pairing:

$$\boxed{\cap : H_{2n-i}(X) \times H_i^{BM}(X) \longrightarrow H_0(X) \simeq \mathbb{C}}$$

<u>Proof.</u> It follows from the cup product map

$$\cup : H_c^{2n-i}(X) \times H^{2n-j}(X) \longrightarrow H_c^{4n-i-j}(X)$$

Together with the Poincaré duality for the cohomology with compact support: $H_c^{2n-i}(X) \simeq H_i(X)$

□

<u>Remark:</u> $H_i^{BM}(X) \simeq H_{2n-i}(X)^* \simeq H_c^i(X)^*$.

▶ Künneth formula

$\exists$ a natural isomorphism: $\boxtimes : H_i^{BM}(X_1) \otimes H_j^{BM}(X_2) \xrightarrow{\sim} H_{i+j}^{BM}(X_1 \times X_2)$

▶ Pull back

Definition Let R be a commutative, Noetherian ring with unit. A sequence $a_1,\dots,a_r$ of elements of R is called a regular sequence if the ideal I generated by $a_1,\dots,a_r$ is a proper ideal of R and the image of a_i in $R/(a_1,\dots,a_{i-1})$ is a non-zero-divisor for $i=1,\dots,r$.

Example: $x_1,\dots,x_n \in \mathbb{C}[x_1,\dots,x_n]$ form a regular sequence.

Definition A closed embedding $i: X \to Y$ of complex algebraic varieties is a regular embedding of codimension d if $\forall x \in X$ $\exists$ an affine neighborhood $U \subset Y$ such that if $I \subset \mathbb{C}[U]$ is the ideal defining $X \cap U$, then I is generated by a regular sequence of length d.

Remark

1. Let J_X be the ideal sheaf of X in Y. Then the normal sheaf $N_X Y := \left(J_X / J_X^2\right)^{\vee}$ is a rank d vector bundle.
2. Let X and Y be smooth varieties. Then a closed embedding $i: X \to Y$ is regular.

Definition A morphism $f: X \to Y$ between complex algebraic varieties is flat if for $U \subset Y, U' \subset X$ affine open sets with $f(U') \subset U$, the induced map $f^*: \mathbb{C}[U] \longrightarrow \mathbb{C}[U']$ makes $\mathbb{C}[U']$ a flat $\mathbb{C}[U]$-module.

Let $f: X \to Y$ be a morphism. One can define the sheaf of relative differentials $\Omega^1_{X/Y}$, which fits into the exact sequence:

$$f^*\Omega^1_Y \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

Definition A morphism $f: X \to Y$ is smooth of relative dimension n if f is flat and of relative dimension n and $\Omega^1_{X/Y}$ is a vector bundle of rank n.

A morphism $f: X \to Y$ is local complete intersection of codimension d if f admits a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "f"'] & Z \arrow[r, "g"] & Y \end{tikzcd}$$

where

- i is a closed regular embedding of codimension n, and
- g is a smooth morphism of relative dimension $d+n$.

Example Any morphism between smooth varieties is l.c.i.

Proposition Let $f: X \longrightarrow Y$ be a lci morphism between complex algebraic varieties of codimension d. Then $\exists$

$$\boxed{f^*: H_i^{BM}(Y) \longrightarrow H_{i+2d}^{BM}(X)}$$

Proof. Use the sheaf theoretic description of BM homology - cf. [CG], Section 8.3.31.

Convolutions in Borel-Moore homology

Pairing

Let X, Y be varieties, embedded as topological closed subsets into an m-dimensional oriented smooth manifold M.

$\exists$ cup-product in relative cohomology:

$$\cup : H^{m-i}(M, M \setminus X) \times H^{m-j}(M, M \setminus Y) \longrightarrow H^{2m-i-j}(M, M \setminus (X \cup Y))$$

it induces the intersection pairing (depending on M):

$$\boxed{\cap : H_i^{BM}(X) \times H_j^{BM}(Y) \longrightarrow H_{i+j-m}(X \cap Y)}$$

Correspondences.

Now, let M_1, M_2 be smooth connected varieties. Let $X \subset M_1 \times M_2$ be a subvariety such that

$$\boxed{\text{the projection } X \longrightarrow M_2 \text{ is proper}}$$

We define an operator

$$\boxed{\Theta_X : H_*^{BM}(M_1) \longrightarrow H_*^{BM}(M_2)}$$ — correspondence

depending on the fundamental class $[X]$ of X, in the following way:

p_1 is lci since M_1, $M_1 \times M_2$ are smooth

$$\begin{array}{ccccccc}
H_*^{BM}(M_1) & \xrightarrow{p_1^*} & H_*^{BM}(M_1 \times M_2) & \xrightarrow{\cap [X]} & H_*^{BM}(X) & \xrightarrow{(p_2)_*} & H_*^{BM}(M_2) \\
c & \longmapsto & p_1^*(c) & \longmapsto & p_1^*(c) \cap [X] & \longmapsto & (p_2)_*(p_1^*(c) \cap [X])
\end{array}$$

here, $Y = M_1 \times M_2$

$$\boxed{\Theta_X(c) := (p_2)_*\left(p_1^*(c) \cap [X]\right)}$$

Remark:

1. $\deg \Theta_X(c) = \deg(c) + 2\dim_{\mathbb{C}} X - 2\dim_{\mathbb{C}} M_1$
2. Let $c \in H_*^{BM}(X)$. Similarly, we can define $\Theta_c : H_i^{BM}(M_1) \longrightarrow H^{BM}_{i+\deg(c)-2\dim(M_1)}(M_2)$.

Maulik-Okounkov's idea: Let X be a smooth quasi-projective complex algebraic variety with a $\mathbb{C}^*$-action.

$$\text{Stab} \text{ “}=\text{” } \Theta_L : H_*^{BM}(X^{\mathbb{C}^*}) \longrightarrow H_*^{BM}(X) \quad (\text{equivariantly})$$

$$R \text{ “}=\text{” } \text{Stab}^{-1} \circ \text{Stab}$$

▶ Composition of correspondences

Let M_1, M_2, M_3 be smooth varieties and let $X_{12} \subset M_1 \times M_2$ and $X_{23} \subset M_2 \times M_3$ be subvarieties such that

the projection $X_{12} \longrightarrow M_2$ is proper	and	the projection $X_{23} \longrightarrow M_3$ is proper

Thus $\Theta_{X_{12}}$ and $\Theta_{X_{23}}$ are well-defined, therefore it makes sense to consider $\Theta_{X_{23}} \circ \Theta_{X_{12}}$.

Question: is $\Theta_{X_{23}} \circ \Theta_{X_{12}}$ a correspondence?

Assumption: $\boxed{p_{13} : p_{12}^{-1}(X_{12}) \times p_{23}^{-1}(X_{23}) \longrightarrow M_1 \times M_3 \text{ is proper}}$

Here, $p_{i,j} : M_1 \times M_2 \times M_3 \longrightarrow M_i \times M_j$ denotes the projection, for $i,j \in \{1,2,3\}$

Remark: Note that $p_{12}^{-1}(X_{12}) \times p_{23}^{-1}(X_{23}) = X_{12} \times_{M_2} X_{23} = (X_{12} \times M_3) \cap (M_1 \times X_{23})$

Proposition. Set $X_{12} \circ X_{23} = p_{13}(X_{12} \times_{M_2} X_{23})$. Then $\Theta_{X_{23}} \circ \Theta_{X_{12}} = \Theta_{X_{12} \circ X_{23}}$.

Definition. Define the convolution product:

$$
\begin{aligned}
H_i^{BM}(X_{12}) \times H_j^{BM}(X_{23}) &\longrightarrow H^{BM}_{i+j-2\dim_{\mathbb{C}}(M_2)}(X_{12} \circ X_{23}) \\
(c_{12}\,,\, c_{23}) &\longmapsto c_{12} * c_{23} := (p_{13})_*(p_{12}^* c_{12} \cap p_{23}^* c_{23})
\end{aligned}
$$

Proposition $\Theta_{c_{23}} \circ \Theta_{c_{12}} = \Theta_{c_{12} * c_{23}}$

Exercises:

1. Let $M = M_1 = M_2 = M_3$ and assume that $X_{12}, X_{23} \subset \Delta(M)$, where $\Delta : M \longrightarrow M \times M$ is the diagonal embedding. Compute $X_{12} \circ X_{23}$.

2. Let $M_1 = pt$, $f : M_2 \longrightarrow M_3$ proper morphism. Set $X_{12} := M_1 \times M_2 = M_2$ and $X_{23} = \mathrm{Graph}(f)$.
   - Show that $X_{12} \circ X_{23} = \mathrm{Im} f$.
   - Let $c \in H_*^{BM}(M_2) = H_*^{BM}(X_{23})$. Show that $c * [\mathrm{Graph}(f)] = f_*(c)$.

3. Let $f : M_1 \longrightarrow M_2$ be a smooth morphism and set $X_{12} := \mathrm{Graph}(f) \subset M_1 \times M_2$, $M_3 = \{pt\}$, $X_{12} = M_2$.
   - Show that $X_{12} \circ X_{23} = M_1$.
   - Let $c \in H_*^{BM}(M_2) = H_*^{BM}(X_{12})$. Show that $[\mathrm{Graph}(f)] * c = f^*(c)$.

Final remark: The constructions described before work for any homology theory for which one has proper pushforwards, pullback w.r.t. l.c.i. morphisms, and fundamental classes, e.g.:

G_0-theory, Chow groups, elliptic cohomology

$$\begin{array}{l} \shortparallel \\ G_0(X) = \{ [\mathcal{F}] : \mathcal{F} \text{ coherent sheaf on } X \} / \sim \end{array}$$