Equivariant (co) homology

Let G be a linear algebraic group and let X be a G-variety with action σ.
Let $\Psi := (\sigma, pr_X) : G \times X \longrightarrow X \times X$.

<u>Definition</u> The action σ is proper if Ψ is proper, and free if Ψ is a closed embedding.

<u>Definition</u> Let X be a G-variety with G-action σ. A morphism $\pi : X \to Y$ of varieties is a geometric quotient of X by G if:

- $\pi \circ \sigma = \pi \circ pr_X$ (i.e., $\pi(gx) = \pi(x)$ $\forall x \in X, \forall g \in G$; $gx := \sigma(g,x)$);
- π is surjective;
- $\Psi(G \times X) = X \times_Y X$;
- $U \subset Y$ open $\Leftrightarrow$ $\pi^{-1}(U)$ open;
- $\mathcal{O}_Y$ is the subsheaf of $\pi_* \mathcal{O}_X$ consisting of invariant functions.
  $(\mathcal{O}_Y \to \pi_* \mathcal{O}_X)$

<u>Definition</u>

We say that $\{(V_i, U_i)\}_{i \in \mathbb{N}}$ is a good system of representations for G if each V_i is a G-representation, $U_i \subseteq V_i$ is a G-invariant open subset satisfying the following conditions:

- G acts freely on U_i and $\exists$ a geometric quotient U_i / G;
- For each i, $\exists$ a G-representation W_i so that $V_{i+1} = V_i \oplus W_i$;
- $U_i \subseteq U_{i+1}$ and the inclusion factors as $U_i = U_i \oplus \{0\} \subseteq U_i \oplus W_i \subseteq U_{i+1}$;
- $\lim_{i \to +\infty} \dim V_i = +\infty$;
- $\operatorname{codim}_{V_i} (V_i \setminus U_i) < \operatorname{codim}_{V_j} (V_j \setminus U_j)$ for $i < j$.

Example: let $G = T = \mathbb{C}^*$. We can choose: $V_i = \mathbb{C}^i$, $U_i = \mathbb{C}^i \setminus \{0\}$, the T-action is given by: $g \cdot (a_1, \dots, a_i) = (ga_1, \dots, ga_i)$

Then $W_i = \mathbb{C}$, T acts freely on U_i and $U_i / T \simeq \mathbb{P}^{i-1}$

Lemma (Heller-Malagon-Lopez, Equivariant Algebraic Cobordism)
Let X be a G-variety and let $\{(V_i, U_i)\}$ be a good system of representations. Then

- for each i $\exists$ $X \times U_i / G =: X \times^G U_i$;
- —"— $X \times^G U_i$ is quasi-projective;
- the morphisms $\phi_{ij} : X \times^G U_i \longrightarrow X \times^G U_j$ is l.c.i.

Theorem (Heller-Malagon-Lopez, Equivariant Algebraic Cobordism)
Let G be a linear algebraic group. Let $\{(V_i, U_i)\}$ be a good system of representations. For X a G-variety, the i-th G-equivariant Borel-Moore homology group of X is:

$$\boxed{H_i^G(X) := \lim_{j} H^{BM}_{i - 2\dim G + 2\dim U_i}(X \times^G U_i)}$$

Then:

1. $H_i^G(X)$ does not depend on $\{(V_i, U_i)\}$;
2. $H_*^G(-)$ defines a homology theory with the same properties of the usual BM homology theory (in the category of G-varieties).

For X also smooth (and equidimensional), the G-equivariant cohomology of X is:

$$\boxed{H_G^*(X) := \lim_{J} H^*(X \times^G U_i)}$$

$T = \mathbb{C}^*$ $\quad BT = \mathbb{P}^\infty_{\mathbb{C}}$

$\uparrow$

$ET = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^\infty}(-1))$

$H_T^*(X) = H^*(X \times_T ET)$

Statements similar to (1) and (2) hold in this case.

Remark: the projective system $\{U_i/G\}$ represents a classifying space of G. Since the action of G on U_i is free, $U_i \longrightarrow U_i/G$ is a principal G-bundle for each G.

Example Let $G = T = (\mathbb{C}^*)^n$. Then $\lim_{J} H^*(U_i/\mathbb{C}^*) = \lim_{J} H^*(\mathbb{P}^J)$

$$\boxed{H_T^2(pt) \simeq t^*} \implies \boxed{H_T^*(pt) \simeq \mathrm{Sym}(t^*) \simeq \mathbb{C}[\varepsilon_1, \dots, \varepsilon_n]}$$

where t = Lie algebra of T, $\varepsilon_i = d\chi_i \in t^*$ is the differential of $\chi_i : T \longrightarrow \mathbb{C}^*$ - i-th character.

Remark: pullback with respect to $p: X \to pt$ (X smooth) induces a graded module structure on $H_G^*(X)$ over $H_G^*(pt)$. (Also, $H_*^G(X)$ is a graded module over $H_G^*(pt)$.)

▶ Restricting the action and induction

Let $H \subseteq G$ be a closed normal subgroup of G.

Facts:

1. The action of H on U_i is free, $U_i/H = (G/H \times U_i)/G =: G/H \times^G U_i$ exists
$\Rightarrow \{(V_i, U_i)\}$ is a good system for H;
2. $X \times^H U_i \longrightarrow H \times^G U_i$ is smooth

Thus, we have

$\boxed{res_{G,H} : H_*^G(X) \longrightarrow H_*^H(X)}$ (and, if X is smooth, $\boxed{res_{G,H} : H_G^*(X) \longrightarrow H_H^*(X)}$)

In particular, for $H = \langle e_G \rangle$:

$\boxed{res_G : H_*^G(X) \longrightarrow H_*^{BM}(X)}$ (and, if X is smooth, $\boxed{res_G : H_G^*(X) \longrightarrow H^*(X)}$)

Example: $res_T : H_T^*(pt) \longrightarrow H^*(pt)$ "sends" $\varepsilon_i \longmapsto 0$ for all i.

Theorem (Heller-Malagon-Lopez, Equivariant Algebraic Cobordism)
Let H be a closed normal subgroup of G. Consider G as an H-scheme with the action $(h,g) \mapsto gh^{-1}$.
Let X be a G-variety. Then

$$\boxed{H_*^H(X) \simeq H_*^G(X \times^H G)}$$

$X \times G/H$

▶ Geometric quotients

Definition

Let X be a G-variety and assume that X/G exists. We say that $\pi : X \to X/G$ is a principal G-bundle over X/G if π is faithfully flat and $\Psi : G \times X \xrightarrow[(a, pr_X)]{} X \times_{X/G} X$ is an isomorphism.

(surjective + flat)

Remark.

1. This definition is equivalent to the condition that π is a locally isotrivial fibration with fiber G.

Let X be a G-variety and assume that X/G exists. Let $\{(V_i, U_i)\}$ be a good system of representations. One can prove that $\pi_i : X \times^G U_i \longrightarrow X/G$ is a l.c.i. morphism for each i. Thus $\exists$

$$\boxed{\pi^* : H_*^{BM}(X/G) \longrightarrow H^G_{*+\dim G}(X)}$$

Proposition (Heller-Malagon-Lopez, Equivariant Algebraic Cobordism)

1. Assume that $\pi : X \to X/G$ is a principal G-bundle. Then π^* is an isomorphism.
2. —— " —— the G-action is proper. —— " ——

Here, we work with $\mathbb{C}$-coefficients. With $\mathbb{Z}$-coefficients, only (1) works; with $\mathbb{Q}$ - —"—, also (2)

▶ Reduction to a torus action

Let G be a connected reductive linear algebraic group.

Let $T \subset G$ be a maximal torus, let $T \subset B$ be a Borel subgroup.

Let N be the normalizer of T in G and $W = N/T$ the Weyl group.

Facts:

1. $\exists\ W \curvearrowright H_*^T(X)$ action
2. $\exists$ map $H_*^G(X) \longrightarrow H_*^T(X)^W$

Proposition

$H_*^G(X) \longrightarrow H_*^T(X)^W$ is an isomorphism.

Example Let $X = pt$ and $G = GL(n)$. Let $T = (\mathbb{C}^*)^n$ and $W = \mathfrak{S}_n$. Then

$$\boxed{H^*_{GL(n)}(pt) \simeq H^*_T(pt)^{\mathfrak{S}_n} \simeq \mathbb{C}[\varepsilon_1, \dots, \varepsilon_n]^{\mathfrak{S}_n}}$$

▶ Chern classes

Let X be a G-variety with a σ-action.

Definition We say that a (coherent) sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is G-equivariant if the following conditions hold:

1. $\exists$ a given isomorphism $I : \sigma^* \mathcal{F} \xrightarrow{\sim} pr_X^* \mathcal{F}$;

   (on $G \times X$)

2. Consider

$$\begin{tikzcd}
G\times G\times X \arrow[r, "m\times id_X"] \arrow[d, "id_G\times\sigma"'] \arrow[dr, phantom, "\circlearrowleft"] & G\times X \arrow[d, "\sigma"] \\
G\times X \arrow[r, "\sigma"] & X
\end{tikzcd}$$

, $p_{23}: G\times G\times X \longrightarrow G\times X$ the projection

and

$$\begin{tikzcd}
(m\times id_X)^*\sigma^*\mathcal{F} \arrow[rr, "(m\times id_X)^*I", "\sim"'] \arrow[d, phantom, "\cong" rotate=90] & & (m\times id_X)^* pr_X^*\mathcal{F} \arrow[dd, no head, "\cong"'] \\
(id_G\times\sigma)^*\sigma^*\mathcal{F} \arrow[r, "\sim", "(id_G\times\sigma)^*I"'] & (id_G\times\sigma)^* pr_X^*\mathcal{F} \arrow[d, phantom, "\cong" rotate=90] & \\
 & (p_{23}^*)^*\sigma^*\mathcal{F} \arrow[r, "\sim", "p_{23}^*I"'] & p_{23}^* pr_X^*\mathcal{F}
\end{tikzcd}$$

The equation $(p_{23}^* I)\circ(id_G\times\sigma)^* I = (m\times id_X)^* I$ holds;

3. Consider $\{e_G\}\times X \xrightarrow[\phi_e]{} G\times X$: $\phi_e^* I = id : \mathcal{F} \simeq \phi^*\sigma^*\mathcal{F} \xrightarrow{\sim} \phi^* pr_X^*\mathcal{F} \simeq \mathcal{F}$

Example: the structure sheaf $\mathcal{O}_X$ is G-equivariant w.r.t. natural isomorphisms: $\sigma^*\mathcal{O}_X \simeq \mathcal{O}_{G\times X} \simeq pr_X^*\mathcal{O}_X$

Exercise: Find the correct notion of equivariance for a vector bundle E for which:

E is G-equivariant $\Leftrightarrow$ its sheaf of section $\mathcal{F}$ is G-equivariant

Proposition (Chern classes)

Let X be a smooth G-variety and let E be a G-equivariant vector bundle on it. Then

$$\boxed{\exists \quad c_i^G(E) \in H_G^*(X) \quad \text{for } i=1,\dots,\operatorname{rank}(E)}$$

Sketch of the proof: $c_i^G(E) = \lim c_i(E_J)$, where $E_J := E \times^G U_i$ is a vector bundle on $X \times^G U_i$. □

Definition Let E be a rank r G-equivariant vector bundle on a G-variety X. We call equivariant Euler class $e^G(E)$ of E the top Chern class $c_r^G(E)$.

Example: Let $X = pt$ and $G = (\mathbb{C}^*)^n$. For any character

$$\boxed{\rho_i : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^*, \ (t_1,\dots,t_n) \longmapsto t_i}$$

we have a 1-dimensional vector space $\mathbb{C}_{\rho_i}$ with a G-action given by ρ_i.
It corresponds to a G-equivariant line bundle L_i on X. Then

$$\boxed{c_1(L_i^\vee) = \varepsilon_i \in H_T^*(pt) \simeq \mathbb{C}[\varepsilon_1,\dots,\varepsilon_n]}$$

▶ Atiyah - Bott localization theorem

Let $G = T = (\mathbb{C}^*)^n$ and let X be a smooth T-variety.

Lemma [Lemma 2.3, Iversen, A fixed point formula for action of tori on algebraic varieties]
The fixed point locus X^T is smooth

Denote by $Z_1, \dots, Z_m$ the connected components of X^T, which are smooth.
Let $i_k : Z_k \longrightarrow X$ be the inclusion, which is a regular closed embedding.
Let N_j be the normal bundle of Z_k in X. It is a T-equivariant vector bundle and

$$\boxed{e^T(N_k) \in H_T^*(Z_k)}$$

Note that there exist

proper pushforward $\boxed{i_{k*} : H_T^*(Z_k) \longrightarrow H_T^*(X)}$ and pullback $\boxed{i_k^* : H_T^*(X) \longrightarrow H_T^*(Z_k)}$

Lemma [Corollary 2.6.44, CG]
For any $\alpha \in H_T^*(Z_k)$, we have $i_k^* \circ i_{k*}(\alpha) = \alpha \cup e^T(N_k)$

Remark: $H_T^*(X)$ and $H_T^*(Z_k)$ are modules over $R_T := H_T^*(pt) = \mathbb{C}[\varepsilon_1, \dots, \varepsilon_n]$.

Let $K_T := \mathbb{C}(\varepsilon_1, \dots, \varepsilon_n)$ be the field of fractions of $H_T^*(pt)$. It makes sense to consider:

$$\boxed{H_T^*(X) \otimes_{R_T} K_T} \quad \text{and} \quad \boxed{H_T^*(Z_k) \otimes_{R_T} K_T}$$

Lemma [Atiyah-Bott, The moment map and equivariant cohomology]

$e^T(N_k)$ is invertible in $H_T^*(Z_k) \otimes_{R_T} K_T$.

Theorem [Atiyah-Bott, The moment map and equivariant cohomology]

Let $i : X^T \longrightarrow X$ be the inclusion. Then

$$\boxed{i_* : H_T^*(X^T) \otimes_{R_T} K_T \simeq \bigoplus_{k=1}^{m} H_T^*(Z_k) \otimes_{R_T} K_T \xrightarrow{\sim} H_T^*(X) \otimes_{R_T} K_T}$$

given by

$$\boxed{(\alpha_k)_k \overset{i_*}{\longmapsto} \sum_k i_{k*}(\alpha_k)}$$

is an isomorphism, with inverse given by

$$\boxed{\alpha \overset{(i_*)^{-1}}{\longmapsto} \left(\frac{i_k^*(\alpha)}{e^T(N_k)} \right)_k}$$

Example: Let $T=\mathbb{C}^*$ and $X=\mathbb{C}$. Then $X^{\mathbb{C}^*}=\{\text{origin}\}$ and $N\simeq T_0\mathbb{C}\simeq\mathbb{C}_\rho$. Then

$$e^{\mathbb{C}^*}(N)=-\varepsilon$$

Let $T=(\mathbb{C}^*)^2$ and $X=\mathbb{C}^2$. Then $X^T=\{\text{origin}\}$ and $N\simeq T_0\mathbb{C}^2\simeq\mathbb{C}_{\rho_1}\oplus\mathbb{C}_{\rho_2}$.

$$e^T(N)=\varepsilon_1\varepsilon_2$$

Let D_x and D_y be the x-axis and the y-axis of $\mathbb{C}^2$, respectively. Then by localization, we get:

$$(i_*)^{-1}\big([D_x]\big)=\frac{i^*([D_x])}{e^T(T_0\mathbb{C}^2)}=\frac{i^*(i_{x*}([D_x]))}{e^T(T_0\mathbb{C}^2)}$$

by abuse of notation, $[D_x]\in H_*^T(\mathbb{C}^2)$ denotes $i_{x*}([D_x])$, where $[D_x]\in H_*^T(D_x)$ is the fundamental class

$$=\frac{j_x^*i_x^*(i_{x*}([D_x]))}{e^T(T_0\mathbb{C}^2)}=\frac{j_x^*\big([D_x]\cup e^T(N_{D_x/\mathbb{C}^2})\big)}{e^T(T_0\mathbb{C}^2)}=\frac{[0]\,\varepsilon_2}{\varepsilon_1\varepsilon_2}$$

$$\begin{tikzcd} \{0\}=(\mathbb{C}^2)^T \arrow[rr, "i"] \arrow[dr, "j_x"'] & & \mathbb{C}^2 \\ & D_x \arrow[ur, "i_x"'] & \end{tikzcd}$$

$$0\to T_{D_x}\to i_x^*T_{\mathbb{C}^2}\to N_{D_x/\mathbb{C}^2}\to 0$$

Exercise:

1. Let $T=(\mathbb{C}^*)^n$ and $X=\mathbb{C}^n$. Compute $c_i^T(N)$, where N is the normal bundle of X^T in X.

Remark:

Assume that X^T is proper. We can define the equivariant integral as:

$$\boxed{\int_X \alpha = \sum_{k=1}^{m} \int_{Z_k} \frac{i_k^*(\alpha)}{e^T(N_k)} \qquad \text{for } \alpha \in H_T^*(X) \otimes_{R_T} K_T}$$

Bialynicki-Birula decomposition

Let us start with the following proposition.

Proposition (Math.stackexchange.com/questions/1540201)

Let X be a complex variety. X is proper $\iff$ $\forall$ smooth proper curve C and $\forall\, U \subseteq C$ open, any morphism $f: U \longrightarrow X$ can be extended (uniquely) to a morphism $C \to X$.

Let X be a smooth projective complex variety with a $\mathbb{C}^*$-action σ.

Fix $x \in X$ and consider the morphism

$$\boxed{\begin{aligned} \sigma_x : \mathbb{C}^* &\longrightarrow X \\ t &\longmapsto \sigma(t,x) =: t\cdot x \end{aligned}}$$

By the lemma above, we get

$$\begin{tikzcd} \mathbb{C}^* \arrow[rr, "\sigma_x"] \arrow[dr, hook] & \circlearrowleft & X \\ & \mathbb{P}^1 \arrow[ur, "\overline{\sigma_x}"'] & \end{tikzcd}$$

Define:

$$\boxed{\lim_{t \to 0} t \cdot x =: \overline{\sigma_x}(0) \in X^{\mathbb{C}^*} \quad \text{and} \quad \lim_{t \to \infty} t \cdot x =: \overline{\sigma_x}(\infty) \in X^{\mathbb{C}^*}}$$

Remark:

1. [CG, Lemma 2.4.1] concerns the existence of $\overline{\sigma_x}$. The proof is based on the analysis of $\overline{\mathrm{Graph}(\sigma_x)}$.
2. Note that $X^{\mathbb{C}^*} \neq \phi$.

For any connected component Z of $X^{\mathbb{C}^*}$, define the subsets:

$$\mathrm{Attr}(Z) := \left\{ x \in X : \lim_{t \to 0} t \cdot x \in Z \right\} \quad \text{stable/attracting variety}$$

$$\mathrm{Rep}(Z) := \left\{ x \in X : \lim_{t \to \infty} t \cdot x \in Z \right\} \quad \text{unstable variety}$$

Remark: Carrell-Sommese and Atiyah-Bott studied (un)stable manifolds associated with a symplectic manifold with the action of a compact torus. When the manifold is Kahler, these manifolds correspond to the above ones with respect to the action of the complexification of the torus

– see [CG, Chapter 2]

Definition. A flat morphism $\phi : Y \longrightarrow Z$ between complex varieties is called an affine fibration of relative dimension d if $\forall\, z \in Z$ $\exists$ a Zariski open neighborhood $U \subset Z$ such that there exists isomorphism

$$\begin{tikzcd} \phi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "\phi"'] & \arrow[d, phantom, "\circlearrowleft"] & U \times \mathbb{C}^d \arrow[dl, "pr_Z"] \\ & U & \end{tikzcd}$$

The following is the main characterization of stable varieties.

Theorem (Białynicki-Birula – 1973, 1976)

1. For any connected component Z of $X^{\mathbb{C}^*}$, $\mathrm{Attr}(Z)$ is smooth and locally closed, and

$$\boxed{\begin{aligned} \lim : \mathrm{Attr}(Z) &\longrightarrow Z \\ x &\longmapsto \lim_{t \to 0} t \cdot x \end{aligned}}$$

is an affine fibration.

2. The relative dimension of the affine fibration $\lim$ is the dimension of $(T_z X)^+$, where z is an arbitrary point of Z and $(T_z X)^+$ appears in the $\mathbb{C}^*$-decomposition:

$$T_z X = \underbrace{(T_z X)^-}_{\{v \in T_z X \,:\, t \cdot v = t^m v \text{ for } m \in \mathbb{Z},\, m < 0\}} \oplus \overbrace{(T_z X)^0}^{= T_z Z = T_z X^{\mathbb{C}^*}} \oplus \underbrace{(T_z X)^+}_{\{v \in T_z X \,:\, t \cdot v = t^m v \text{ for } m \in \mathbb{Z},\, m > 0\}}$$

3. For any $z \in Z$, $T_z \operatorname{Attr}(Z) = (T_z X)^{\circ} \oplus (T_z X)^{+}$.

4. X is the union of the $\operatorname{Attr}(Z)$'s, by varying of the connected components Z of $X^{\mathbb{C}^*}$.

5. There exist an ordering $X^{\mathbb{C}^*} = \bigsqcup_{k=1}^{m} Z_k$ of the connected components of the fixed point locus and a finite decreasing sequence of closed subvarieties of X such that:

$$X = X_m \supset X_{m-1} \supset \cdots \supset X_1 \supset X_0 = \phi$$

such that $X_i \setminus X_{i-1} = \operatorname{Attr}(Z_i)$.

Remark: $\exists$ a version of the theorem above for the unstable varieties.

Example: let $X = \mathbb{P}^n$. Consider the $\mathbb{C}^*$-action:

$$\begin{aligned} \sigma : \mathbb{C}^* \times \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ (t, [x_0, \dots, x_n]) &\longmapsto [t^n x_0, t^{n-1} x_1, \dots, t x_{n-1}, x_n] \end{aligned}$$

$(\mathbb{P}^n)^{\mathbb{C}^*}$ consists of the following points: $z_0 := [1, 0, \dots, 0]$, $z_1 := [0, 1, 0, \dots, 0], \dots, z_n := [0, \dots, 0, 1]$

Thanks to Bialynicki-Birula theorem, $Attr(z_i)$ is an affine space for $i = 0, \dots, n$. Thus we need only to compute $\dim (T_{z_i} \mathbb{P}^n)^+$.

Exercise: verify that the relative dimension of $\lim Attr(z_i) \longrightarrow \{z_i\}$ is i.

Consequences of (5) from the Theorem

Note that

$$Attr(z_1) = X_1 \qquad \text{is closed in } X$$

$$Attr(z_1) \cup Attr(z_2) = X_2 \qquad \text{— '' —}$$

$$\vdots$$

$$Attr(z_1) \cup \dots \cup Attr(z_k) = X_k \qquad \text{— '' —}$$

The subvarieties $Attr(z_i)$'s form a so-called α-partition.

Proposition Assume that $X^{\mathbb{C}^*}$ consists of a finite number of points. Then $H^{BM}_{odd}(X) = 0$.

Proof. Exercise using the α-partition above.

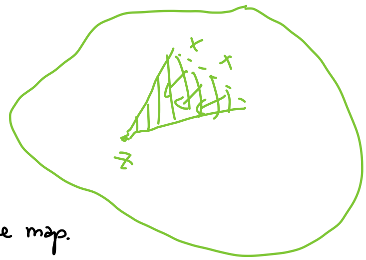
Note that for any $i \in \{1, \dots, m\}$, $Attr(z_i) = X_i \setminus X_{i-1} \subset X_i$ which is closed in X. Then

$$\boxed{\begin{aligned} \overline{Attr(z_i)} \subset X_i &= (X_i \setminus X_{i-1}) \cup X_{i-1} = (X_i \setminus X_{i-1}) \cup (X_{i-1} \setminus X_{i-2}) \cup X_{i-2} = \dots \\ &= \bigcup_{k \leq i} Attr(z_k) =: Attr^f(z_i) \end{aligned}}$$

Remark:

1. the relation $\leq$ on the indices labelling the connected components of $X^{\mathbb{C}^*}$ is the transitive closure of the relation :

$$\overline{\mathrm{Attr}(Z_i)} \cap Z_k \neq \phi \Rightarrow i \gtrsim k$$



2. Let "$\mathrm{Stab}: H_T^*(X^{\mathbb{C}^*}) \longrightarrow H_T^*(X)$" be the stable envelope map.

Then, for any connected component Z of $X^{\mathbb{C}^*}$ and for any equivariant class γ in Z :

$$\boxed{\mathrm{Stab}(\gamma) = \text{pushforward of a class in } \mathrm{Attr}^f(Z)} \quad \doteq \bigcup_{Z' \leq Z} \mathrm{Attr}(Z')$$

Attention: In the construction of stable envelopes, X is only quasi-projective!