Goal of Today

We shall introduce all the relevant notions and techniques to prove:

Theorem There exists a unique map of $H_T^*(pt)$-modules

$$\boxed{\mathrm{Stab}_{C,\varepsilon} : H_T^*(X^A) \longrightarrow H_T^*(X)}$$

such that for any connected component Z of X^A and any $\gamma \in H^*_{T/A}(Z)$, the stable envelope

$$\boxed{\Gamma := \mathrm{Stab}_{C,\varepsilon}(\gamma)}$$

satisfies:

(i) $\operatorname{supp} \Gamma \subset \mathrm{Attr}_C^f(Z)$;

(ii) $\Gamma|_Z = \pm e(N_-(Z)) \cup \gamma$, where the sign is set to that $\pm e(N_-(Z)) = \varepsilon$ in $H_A^*(pt)$;

(iii) $\deg_A \Gamma|_{Z'} < \frac{1}{2} \operatorname{codim}_{\mathbb{R}} Z'$, for any $Z' <_C Z$.

Maulik-Okounkov's setting

Let G be a reductive (linear algebraic) group and let $A \subset T \subset G$ be a pair of tori.
Denote by $\mathfrak{a} \subset \mathfrak{t} \subset \mathfrak{g}$ the corresponding Lie algebras.

Let X be a smooth connected complex algebraic variety and let $\omega \in H^0(X, \Omega^2_X)$ be an algebraic symplectic form on X.

$\parallel$

(non-degenerate closed)

Assume that:

1. $\exists$ a proper morphism $\pi : X \to X_0$ to an affine variety X_0.

2. $\exists$ actions of G on X and X_0 such that π is G-equivariant. Denote by $\tau : G \times X \to X$ and $\tau_0 : G \times X_0 \longrightarrow X_0$ the actions.

$$\tau_g : \begin{array}{ccc} X & \longrightarrow & X \\ x & \longmapsto & \tau(g,x) =: g\cdot x \end{array}$$

3. $\exists$ a nontrivial character $\rho : G \longrightarrow \mathbb{C}^*$ such that for any $g \in G$, we have $\tau_g^* \omega = \rho(g) \cdot \omega$, and $\rho(a) = 1 \quad \forall\, a \in A$ ($\Rightarrow A$ preserves the symplectic form $\omega \Rightarrow X^A$ is symplectic)

4. X is a formal T-variety.

What is T-formality?

Let $T \simeq (\mathbb{C}^*)^n$. Recall that the torus-equivariant cohomology of a topological space X is

$$H_T^*(X) := H^*(X \times^T ET)$$

where

- $BT = (\mathbb{P}^\infty)^n$,
- $ET = \mathrm{Tot}\left(\pi_1^* \mathcal{O}_{\mathbb{P}^\infty}(-1) \otimes \cdots \otimes \pi_n^* \mathcal{O}_{\mathbb{P}^\infty}(-1)\right)$, where $\pi_i : BT \longrightarrow \mathbb{P}^\infty$ is the i-th projection
- $ET \longrightarrow BT$ is a principal T-bundle
- $X \times^T ET \longrightarrow BT$ is a fiber bundle over BT, with fiber X.

Attention: if you do not know spectral sequences, see Appendix C in Cox-Little-Schenck, Toric varieties

The Serre spectral sequence relative to $X \times^T ET \longrightarrow BT$ has E_2-term:

$$E_2^{p,q} := H_*^p(BT, H_*^q(X; \mathbb{C})) \overset{\text{by universal coefficient theorem}}{\simeq} H^p(BT; \mathbb{C}) \otimes_{\mathbb{C}} H^q(X; \mathbb{C})$$

and it converges to $H^{p+q}(X \times^T ET) = H_T^{p+q}(X)$.

Definition We say that X is *T-formal* if the above spectral sequence degenerates at E_2.

Proposition If $H^q(X;\mathbb{C})=0$ for odd q, X is T-formal.

Proof Since $H^*(BT)\simeq \mathbb{C}[\varepsilon_1,\dots,\varepsilon_n]$ with $\deg(\varepsilon_i)=2$, we have $H^p(BT)=0$ for odd p. Since the differential of the spectral sequence is:

$$d_r^{p,q}: E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

we get $d_r=0$ $\forall r\geq 2$. $\square$

Set $R_T := H^*(BT;\mathbb{C}) = H_T^*(pt)$.

Corollary Let X be a T-formal space. Then

1. there exists an isomorphism of R_T-modules:

$$H_T^*(X)\simeq R_T\otimes_{\mathbb{C}} H^*(X)$$

In particular, $H_T^*(X)$ is a free R_T-module.

2. Assume that X^T consists of a finite number of points, set $n=\#X^T$. Let $i: X^T\longrightarrow X$ be the inclusion. Then

$$i^*: H_T^*(X)\longrightarrow H_T^*(X^T)\simeq R_T^{\oplus n}$$

is injective.

Proof.

1. (Exercise) Use the Leray-Hirsch isomorphism.
2. (Exercise) Atiyah-Bott localization theorem implies that $\mathrm{Ker}(\iota^*)$ is a torsion R_T-module.

Thanks to (1), the assertion follows. $\square$

Back to our setting: the varieties we consider have zero odd cohomology.

Let X be a smooth connected complex algebraic variety and let $\omega \in H^0(X, \Omega_X^2)$ be an algebraic symplectic form on X.

Assume that:

1. $\exists$ a proper morphism $\pi : X \longrightarrow X_0$ to an affine variety X_0.
2. $\exists$ actions of G on X and X_0 such that π is G-equivariant.
3. $\exists$ a nontrivial character $\rho : G \longrightarrow \mathbb{C}^*$ such that for any $g \in G$, we have $\tau_g^* \omega = \rho(g) \cdot \omega$, and $\rho(a) = 1$ $\forall a \in A$ ($\Rightarrow$ A preserves the symplectic form $\omega \Rightarrow X^A$ is symplectic)
4. X is a formal T-variety.

Attention: Okounkov stated that condition (4) could be removed.

Notation: we denote by $\hbar \in \mathfrak{g}^*$ the G-weight of ω, i.e., $\hbar = d\rho$.

Chamber decomposition

Recall that the set of characters $\lambda : A \longrightarrow \mathbb{C}^*$ form a free abelian group (= lattice) of rank the rank of A. Denote it by $\mathrm{Char}(A)$.

Similarly, ——— " ——— cocharacters $\sigma : \mathbb{C}^* \longrightarrow A$ ——— " ——— rank of A. Denote it by $\mathrm{Cochar}(A)$.

Recall that $\exists$ a pairing:

$$\begin{aligned} \langle \cdot , \cdot \rangle : \mathrm{Cochar}(A) \times \mathrm{Char}(A) &\longrightarrow \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) \simeq \mathbb{Z} \\ (\sigma, \lambda) &\longmapsto \lambda \circ \sigma \end{aligned}$$

Set

$$\mathfrak{a}_{\mathbb{R}} := \mathrm{Cochar}(A) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^{\mathrm{rk}(A)}$$

Remark: Note $\exists$ inclusion $\mathfrak{a}_{\mathbb{R}} \hookrightarrow \mathfrak{a}$.

Let $\chi \in \mathfrak{a}^*_{\mathbb{R}}$. We define

$$\chi^{\perp} := \{ v \in \mathfrak{a}_{\mathbb{R}} \mid \chi(v) = 0 \}$$

Let $N(X^A)$ be the normal bundle of X^A in X. It is an A-equivariant bundle over X^A. Thus

$$N(X^A) \simeq \bigoplus_{\lambda \in \Delta} N_\lambda$$

where

1. $\Delta \subset Char(A)$

2. N_λ is the subbundle of $N(X^A)$ whose fibers are the eigenspaces of the A-action corresponding to λ.

Definition We call Δ the *set of roots* of A.

Equivalently, Δ is the set of weights $\alpha = d\lambda \in \mathfrak{a}^*$, where λ appears in the decomposition of $N(X^A)$.

Note that Δ can be seen as a subset of $\mathfrak{a}^*_{\mathbb{R}} \simeq Char(A) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition. We call $\alpha^\perp \subset \mathfrak{a}_{\mathbb{R}}$ the *root hyperplane* associated with $\alpha \in \Delta$.

Fact: the complement of the union of the root hyperplanes $\alpha^\perp$ for $\alpha \in \Delta$ admits a partition into open subsets C_i, called *chambers*, which are its connected components:

$$\mathfrak{a}_{\mathbb{R}} \setminus \left(\bigcup_{\alpha \in \Delta} \alpha^\perp \right) = \bigsqcup_i C_i$$

Let $\sigma \in Cochar(A)$. It induces a $\mathbb{C}^*$-action on X by $(t,x) \longmapsto \tau(\sigma(t),x)$.
We denote by X^σ the fixed point locus of this $\mathbb{C}^*$-action.

Fact: If $\sigma \notin \alpha^\perp$ for any $\alpha \in \Delta$, $X^\sigma = X^A$.

Definition Let C be a chamber. A point $x \in X$ is said to be C-stable if for any $\sigma \in C$ a co-character, the limit $\lim_{t \to 0} \sigma(t) \cdot x$ exists.

Fact: If $\lim_{t \to 0} \sigma(t) \cdot x$ exists, it belongs to X^A and it does not depend on the choice of $\sigma \in C$.

Notation: We denote $\lim_{t \to 0} \sigma(t) \cdot x$ for $\sigma \in C$ by $\lim_C(x)$.

Let Z be a connected component of X^A. Consider the subset

$$\mathrm{Attr}_C(Z) = \{ x \in X : x \text{ is } C\text{-stable and } \lim_C(x) \in Z \}$$

Note that the A-action on X can be "linearized", i.e., the following holds:

Theorem (Sumihiro, Equivariant completion)
Let G be a connected linear algebraic group and let X be a smooth quasi-projective G-variety. Then $\exists$ a finite-dimensional vector space V, a representation $\rho : G \longrightarrow GL(V)$ and an embedding $\phi : X \longrightarrow \mathbb{P}(V)$, which is equivariant in the following sense:

$$\phi(g \cdot x) = \rho(g) \cdot \phi(x) \quad \forall x \in X, \forall g \in G$$

Thus, we have the following:

Theorem

$\mathrm{Attr}_C(Z)$ is smooth and locally closed, and the following map is an affine fibration:

$$\boxed{\begin{aligned} \lim : \mathrm{Attr}_C(Z) &\longrightarrow Z \\ x &\longmapsto \lim_C(x) \end{aligned}}$$

Partial order by attraction

Fix a chamber C. $\exists$ a partial ordering $\leq_C$ on the set of connected components of X^A defined as the transitive closure of the relation:

$$\boxed{\overline{\mathrm{Attr}_C(Z)} \cap Z' \neq \phi \Rightarrow Z' \leq_C Z}$$

Lemma For any connected component Z of X^A, the following set is closed:

$$\boxed{\mathrm{Attr}_C^f(Z) := \bigsqcup_{Z' \leq_C Z} \mathrm{Attr}_C(Z')}$$

Proof.

Consider the proper morphism $\pi : X \longrightarrow X_0$ with X_0 an affine subvariety.

There is an induced A-action on X_0 such that the map π is A-equivariant.

Let $X_0 \longrightarrow V$ be an A-equivariant embedding into a linear representation V of A.
(see [Proposition 1.9, Brion, Introduction to actions of algebraic groups]).

Let $\sigma \in C$ and consider the induced $\mathbb{C}^*$-action on V. Set:

$$V_{\geq 0} := \operatorname{span}\left\{ v \in V : t \cdot v = t^m v \quad \text{for } m \geq 0 \right\}$$

Fact: $\pi\left(\overline{\operatorname{Attr}_C(Z)}\right) \subset X_0 \cap V_{\geq 0}$
(This follows from the characterization of orbit closures - see [Proposition 1.11, Example 1.12, B]).

In particular, for $x \in \overline{\operatorname{Attr}_C(Z)} \Rightarrow \pi(x) \in V_{\geq 0} \Rightarrow \exists \lim_{t \to 0} \sigma(t) \cdot \pi(x)$

The properness of π implies that $\exists\ z' := \lim_{t \to 0} \sigma(t) \cdot x \in \overline{\operatorname{Attr}_C(Z)} \cap X^A$.

Let Z' be the connected component of X^A which contains z'. Then

$$\boxed{z' \in \overline{\operatorname{Attr}_C(Z)} \cap Z' \Rightarrow Z' \leq_C Z} \quad \text{and} \quad \boxed{x \in \overline{\operatorname{Attr}_C(Z)} \Rightarrow x \in \operatorname{Attr}_C(Z')}$$

$\square$

Example

$X = T^* \mathbb{P}^n$. Consider

$$\boxed{(\mathbb{C}^*)^n \simeq A \subset T = A \times \mathbb{C}^*}$$

action on the fibers with weight $-\hbar$

where the A-action on X is induced from the action on $\mathbb{C}^{n+1}$ given by characters denoted by $u_1, \ldots, u_n$.

We identify $\mathbb{P}^n$ with its zero section inside X. Then

$$X^A = \{ p_0 = [1, 0, \ldots, 0], p_1 = [0, 1, 0, \ldots, 0], \ldots, p_n = [0, \ldots, 0, 1] \}$$

Set $n=1$ and $u := u_1$. For $t \in A \simeq \mathbb{C}^*$ and $[x,y] \in \mathbb{P}^1$, the A-action is:

$$t \cdot [x,y] = [t^u x, y]$$

We have 2 roots:

$$\Delta = \{\alpha, -\alpha\} \subset \mathfrak{a}^*_{\mathbb{R}} \simeq \mathbb{R} \quad \text{with} \begin{cases} \alpha(t) = t^u \\ -\alpha(t) = t^{-u} \end{cases}$$

Q. Does Δ resemble some set appearing in Lie Theory? $\Delta \leftrightarrow sl(2)$

Then

$$\boxed{\mathfrak{a}_{\mathbb{R}} \simeq \mathbb{R} = C_+ \sqcup \alpha^{\perp} \sqcup C_-}$$

where

$$\boxed{C_{\pm} = \left\{ x \in \mathbb{R} : \langle \pm\alpha, x \rangle = u \cdot x > 0 \right\}} \quad \text{and} \quad \boxed{\alpha^{\perp} = \{0\}}$$

Now

$$\lim_{C_+}(x) = \lim_{t \to 0} \sigma(t).x = \lim_{t \to 0} \left(\left[t^{u \cdot x} x, y \right], v \right) = p_0 = [1, 0] \quad \forall \Rightarrow$$ To be completed

$$\lim_{C_+}(x) = \lim_{t \to 0} \sigma(t).x = \lim_{t \to 0} \left(\left[t^{u \cdot x} x, y \right], v \right) = p_1 = [0, 1] \quad \forall \Rightarrow$$ To be completed

$$\lim_{C_-}(x) = \lim_{t \to 0} \sigma(t).x = \lim_{t \to 0} \left(\left[t^{u \cdot x} x, y \right], v \right) = p_0 = [1, 0] \quad \forall \Rightarrow$$ To be completed

$$\lim_{C_-}(x) = \lim_{t \to 0} \sigma(t).x = \lim_{t \to 0} \left(\left[t^{u \cdot x} x, y \right], v \right) = p_1 = [0, 1] \quad \forall \Rightarrow$$ To be completed

Supports

Let $i : Y \hookrightarrow X$ be a closed T-invariant subset.

Recall that we have a long exact sequence in equivariant Borel-Moore homology:

$$\cdots \longrightarrow H_p^T(Y) \xrightarrow{i_*} H_p^T(X) \xrightarrow{J^*} H_p^T(X \setminus Y) \longrightarrow H_{p-1}^T(Y) \longrightarrow \cdots$$

Definition We say that $\gamma \in H_T^*(X)$ is supported on Y if

$$\boxed{J^*(PD(\gamma)) = 0}$$

or, equivalently, there exists a class $\tilde{\gamma} \in H_*^T(Y)$ such that

$$\boxed{i_*(\tilde{\gamma}) = PD(\gamma)}$$

If γ is supported on Y, we use the notation: $\mathrm{Supp}(\gamma) \subset Y$.

Polarization

Let $Z \subset X$ be an invariant closed subvariety. Recall

1. $\exists$ fundamental class $[Z]$,
2. $[Z] = PD(e(N(Z)))$, where $N(Z)$ is the normal bundle of Z in X,
3. $\deg e(N(Z)) = \mathrm{codim}_{\mathbb{R}}(Z)$

Let Z be a connected component of X^A. Fix a chamber C.
Then $\exists$ a decomposition into A-weights that are $\underbrace{\text{positive and negative}}_{\langle \alpha, \sigma \rangle \gtrless 0,\ \sigma \in C}$ on C, respectively:

$$N(Z) = N_+(Z) \oplus N_-(Z)$$

Fact 1: The symplectic form induces: $(N_+(Z))^\vee \simeq N_-(Z) \otimes \hbar$

Trivial line bundle with G-action with weight $\hbar$

$$\left.\begin{array}{r} \text{The character corresponding to the weight } \hbar \text{ is trivial on } A \\ \text{Fact 1} \end{array}\right\} \Rightarrow \text{Fact 2: } \alpha \in \Delta \Rightarrow -\alpha \in \Delta$$

Notation: $i: Y \hookrightarrow X$ closed T-invariant subset. For $\eta \in H_T^*(X)$, we set $\eta|_Y := i^*(\eta)$.

Therefore, the following class is a perfect square:

roots that occur in $N(Z)$

$$\boxed{\varepsilon^2 = (-1)^{\frac{1}{2}\operatorname{codim}_{\mathbb{R}}(Z)}\, e(N(Z))\Big|_{pt} = \prod_{i=1}^{\frac{1}{2}\operatorname{codim}(Z)} \alpha_i^2 \in H_A^*(pt)}$$

Definition A choice of a square root $\varepsilon \in H_A^*(Z)$ is called a polarization of Z in X.
The sign in $\pm e(N_-(Z))$ agrees with polarization if $\pm e(N_-(Z)) \in H_T^*(Z)$ restricts to ε in $H_A^*(pt)$.

Degree in A

Fact: Since the A-action on X^A is trivial, we have

$$\boxed{H_T^*(X^A) \simeq H_{T/A}^*(X^A) \otimes_{\mathbb{C}[\mathfrak{t}/\mathfrak{a}]} \mathbb{C}[\mathfrak{t}]}$$

Definition Let $\gamma \in H_T^*(X^A)$.

The degree $\deg_A(\gamma)$ is the degree of γ in $\mathbb{C}[\mathfrak{a}]$ via the factorization $\mathbb{C}[\mathfrak{t}] \simeq \mathbb{C}[\mathfrak{a}] \otimes \mathbb{C}[\mathfrak{t}/\mathfrak{a}]$

Main Theorem

Fix a chamber C and a polarization ε of X^A.

Theorem There exists a unique map of $H_T^*(pt)$-modules

$$\boxed{\mathrm{Stab}_{C,\varepsilon} : H_T^*(X^A) \longrightarrow H_T^*(X)}$$

such that for any connected component Z of X^A and any $\gamma \in H_{T/A}^*(Z)$, the stable envelope

$$\boxed{\Gamma := \mathrm{Stab}_{C,\varepsilon}(\gamma)}$$

satisfies:

(i) $\mathrm{supp}\, \Gamma \subset \mathrm{Attr}_{\mathfrak{C}}^{f}(Z)$;

(ii) $\Gamma|_Z = \pm e(N_-(Z)) \cup \gamma$, where the sign agrees with the polarization ϵ;

(iii) $\deg_A(\Gamma|_{Z'}) < \frac{1}{2}\operatorname{codim}_{\mathbb{R}}(Z')$, for any $Z' <_{\mathfrak{C}} Z$.