# Recap

Consider $A \subset T$ tori.

## ▶ Chambers

Set

$$\mathfrak{a}_{\mathbb{R}} := \mathrm{Cochar}(A) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^{rk(A)}$$

$$\mathfrak{a}^*_{\mathbb{R}} \simeq \mathrm{Char}(A) \otimes_{\mathbb{Z}} \mathbb{R}$$

Moreover, the pairing $\langle \cdot, \cdot \rangle : \mathrm{Cochar}(A) \times \mathrm{Char}(A) \longrightarrow \mathbb{Z}$ extends

$$\langle \cdot, \cdot \rangle : \mathfrak{a}_{\mathbb{R}} \times \mathfrak{a}^*_{\mathbb{R}} \longrightarrow \mathbb{R}$$

Definition We call a root a character $\lambda \in \mathrm{Char}(A)$ which appears in the decomposition

$$N(X^A) \simeq \bigoplus_{\lambda} N_{\lambda}$$

of the normal bundle $N(X^A)$ of $X^A$ in $X$. Let $\Delta \subset \mathrm{Char}(A)$ be the set of roots.

We have

$$\mathfrak{a}_{\mathbb{R}} \setminus \Big( \bigcup_{\alpha \in \Delta} \alpha^{\perp} \Big) = \bigsqcup_i \underset{\text{chambers}}{C_i}$$

## Polarization

Let $Z \subset X^A$ be a connected component.

<u>Definition</u> The polarization of $Z$ is a formal choice of a sign in:

$$\boxed{\varepsilon\Big|_{pt} = \pm \prod_{i=1}^{\frac{1}{2}\mathrm{codim}(Z)} \overset{\text{roots that occur in } N(Z)}{\alpha_i}}$$

The sign in $\pm e(N_-(Z))$ agrees with polarization if $\pm e(N_-(Z)) \in H_T^*(Z)$ restricts

to $\varepsilon$ in $H_A^*(pt)$.

## Degree in A

<u>Fact</u>: Since the A-action on $X^A$ is trivial, we have

$$\boxed{H_T^*(X^A) \simeq H_{T/A}^*(X^A) \otimes_{\mathbb{C}[t/\mathfrak{a}]} \mathbb{C}[t]}$$

**Definition** Let $\gamma \in H^*_T(X^A)$.

The degree $\deg_A(\gamma)$ is the degree of $\gamma$ in $\mathbb{C}[\mathfrak{a}]$ via the factorization $\mathbb{C}[\mathfrak{t}] \simeq \mathbb{C}[\mathfrak{a}] \otimes \mathbb{C}[\mathfrak{t}/\mathfrak{a}]$

**Example**

Let $X = T^*\mathbb{P}^n$,

$$\boxed{\underbrace{A}_{\text{acting on } \mathbb{P}^n} = (\mathbb{C}^*)^n \subset T = A \times \underbrace{\mathbb{C}^*}_{\text{scaling the fibers with weight } \hbar}}$$

Then

$$H^*_T(T^*\mathbb{P}^n) \underset{\substack{| \\ T^*\mathbb{P}^n \text{ equivariantly retracts to } \mathbb{P}^n}}{\simeq} \mathbb{C}[c, u_1, \dots, u_n, \hbar] / c(c-u_1)\cdots(c-u_n)$$

where $c := c_1(\mathcal{O}_{\mathbb{P}^n}(-1))$.

Recall:

$$X^A = \{p_0 = [1, 0, \dots, 0], \dots, p_n = [0, \dots, 0, 1]\}$$

$$\Downarrow$$

$$H^*_T(X^A) \simeq \mathbb{C}[u_1, \dots, u_n, \hbar]^{\oplus n+1}$$

Then

$$\boxed{\deg_A = \deg_{u_1, \dots, u_n}}$$

# Main Theorem

Fix a chamber $C$ and a polarization $\varepsilon$ of $X^A$.

**Theorem** There exists a unique map of $H_T^*(pt)$-modules

$$\boxed{\mathrm{Stab}_C : H_T^*(X^A) \longrightarrow H_T^*(X)}$$

such that for any connected component $Z$ of $X^A$ and any $\gamma \in H^*_{T/A}(Z)$, the stable envelope

$$\boxed{\Gamma := \mathrm{Stab}_C(\gamma)}$$

satisfies:

$\bigsqcup_{Z' \leq_C Z} \mathrm{Attr}_C(Z') = \mathrm{Attr}_C^f(Z)$

(i) $\mathrm{supp}\, \Gamma \subset \mathrm{Attr}_C^f(Z)$;

"morally", $PD(\mathrm{Stab}_C(-)|_Z) = [\mathrm{Attr}_C^f(Z)] \cap (-)$

(ii) $\Gamma|_Z = \pm e(N_-(Z)) \cup \gamma$, where the sign agrees with the polarization $\varepsilon$;

(iii) $\deg_A(\Gamma|_{Z'}) < \frac{1}{2}\mathrm{codim}_{\mathbb{R}}(Z')$, for any $Z' <_C Z$.

Proof: Unicity

We shall prove the following claim:

Claim: let $\gamma \in H_T^*(X)$ be s.t.

1. $\operatorname{supp}(\gamma) \subset \operatorname{Attr}_C^f(Z)$ for some connected component $Z$ of $X^A$.
2. $\deg_A(\gamma_{|Z'}) < \frac{1}{2}\operatorname{codim}_{\mathbb{R}}(Z')$ $\forall$ connected component $Z'$ of $X^A$.

Then $\gamma = 0$.

Let $i : Z \hookrightarrow X$ be a connected component of $X^A$ such that $\operatorname{supp}(\gamma) \subset \operatorname{Attr}_C^f(Z)$.
Let us factor $i$ as $f_3 \circ f_2 \circ f_1$:

$$\begin{array}{ccccccc} Z & \overset{f_1}{\hookrightarrow} & \operatorname{Attr}_C(Z) & \overset{f_2}{\hookrightarrow} & \operatorname{Attr}_C^f(Z) & \overset{f_3}{\hookrightarrow} & X \\ & | & & | & & | & \\ & \text{regular closed embedding} & & \text{open embedding} & & \text{closed embedding} & \end{array}$$

Moreover, $\exists\, \alpha \in H_*^T(\operatorname{Attr}_C^f(Z))$ s.t. $(f_3)_*(\alpha) = PD(\gamma)$. Then we have

$$\boxed{PD(i^*(\gamma)) = e(N_-(Z)) \cap f_1^* f_2^*(\alpha)}$$

(Attention: $N(\operatorname{Attr}_C^f(Z)) = N_-(Z)$)

Now, $\deg_A e(N_-(Z)) = \frac{1}{2}\operatorname{codim}_{\mathbb{R}}(Z)$ (cf. definition of polarization!), and

$$e(N_-(Z))\cdot \_ : H_T^*(X) \longrightarrow H_T^*(X)$$

is an injective map, since $e(N(Z))$ is invertible in $H_T(X)\otimes_{H_T^*(pt)} \mathrm{Frac}(H_T^*(pt))$ and $X$ is formal.

By hypothesis, $\deg_A i^*(\gamma) < \frac{1}{2}\mathrm{codim}_{\mathbb{R}}(Z)$. Thus

$$\boxed{\deg(f_1^* f_2^*(\alpha)) > 2\dim_{\mathbb{C}} Z \implies f_1^* f_2^*(\alpha) = 0}$$

Since $\mathrm{Attr}_C(Z)$ is an affine fibration over $Z$, $f_1^*$ is an isomorphism. Thus

$$\boxed{f_2^*(\alpha) = 0,\ \text{i.e., } \alpha = 0 \text{ over } \mathrm{Attr}_C(Z) \subset \mathrm{Attr}_C^f(Z) := \bigcup_{Z' \leq_C Z} \mathrm{Attr}_C(Z')}$$

Thus,

$$\boxed{\mathrm{supp}(\gamma) \subset \bigsqcup_{\underline{\underline{Z' <_C Z}}} \mathrm{Attr}_C(Z') = \bigsqcup_{Z'' \leq_C Z'} \mathrm{Attr}_C(Z'') \quad \text{with } Z' <_C Z}$$

By arguing inductively, we get $\gamma = 0$.

Now, if $\Gamma_1$ and $\Gamma_2$ in $H_T^*(X)$ satisfy (i), (ii), (iii), $\gamma := \Gamma_1 - \Gamma_2$ satisfies (1) and (2) $\implies \gamma = 0$ $\square$

To prove the existence, let us recall what a correspondence is.

## Correspondences.

Now, let $M_1, M_2$ be smooth connected varieties. Let $X \subset M_1 \times M_2$ be a subvariety such that

$$\boxed{\text{the projection } X \longrightarrow M_2 \text{ is proper}}$$

Let $c \in H_*^T(X)$. We define an operator

$$\boxed{\Theta_c : H_*^{BM}(M_1) \longrightarrow H_*^{BM}(M_2)}$$ — correspondence

in the following way:

$$\begin{array}{ccccccc} H_T^*(M_1) & \xrightarrow{p_1^*} & H_T^*(M_1 \times M_2) & \xrightarrow{\cap c} & H_*^T(X) & \xrightarrow{(p_2)_*} & H_T^*(M_2) \\ \gamma & \longmapsto & p_1^*(\gamma) & \longmapsto & p_1^*(\gamma) \cap c & \longmapsto & PD\left((p_2)_*(p_1^*(\gamma) \cap c)\right) \end{array}$$

$$\boxed{\Theta_c(\gamma) := PD\left((p_2)_*\left(p_1^*(\gamma) \cap c\right)\right)}$$

**Remark:** $\deg \Theta_c(\gamma) = \deg(\gamma) + \deg(c) - 2\dim_{\mathbb{C}}(M_1)$

**Definition** A T-equivariant cycle is a $\mathbb{Q}$-linear formal combination of fundamental classes of invariant subvarieties:

$$\boxed{L := \sum a_k [Z_k]}$$

Recall that:

- $Z \subset X$ is isotropic is $\omega|_{Z^{sm}} = 0$, where $Z^{sm} = \{ z \in Z : z \text{ smooth pt} \}$
- $Z \subset X$ is Lagrangian if $Z$ is isotropic and $\dim Z = \frac{1}{2} \dim_{\mathbb{C}} X$.

**Definition** A cycle $L = \sum a_k [Z_k]$ is Lagrangian if each $Z_k$ is Lagrangian.

**Lemma** ([Lemma 3.4.5., MO])
Let $L \subset X$ be an A-invariant Lagrangian subvariety such that $L \subset \mathrm{Attr}_{\mathfrak{C}}^f(Z)$ for some connected component $Z$ of $X^A$.
Then $\exists!$ a Lagrangian cycle $L'$ such that

- $\mathrm{Supp}(L' - [L]) \subset \bigsqcup_{\underline{Z' <_{\mathfrak{C}} Z}} \mathrm{Attr}_{\mathfrak{C}}^f(Z')$,
- $\deg_A \left( L'|_{Z'} \right) < \frac{1}{2} \mathrm{codim}_{\mathbb{R}}(Z')$ for any $Z' <_{\mathfrak{C}} Z$.

**Remark:** From the Lemma, we get $L' = [L] + \gamma_L$ such that

1. $L'|_{\mathrm{Attr}_C^f(Z)} = [L]|_{\mathrm{Attr}_C^f(Z)}$
2. $\deg_A(L'|_{Z'}) < \frac{1}{2}\mathrm{codim}_{\mathbb{R}}(Z')$ for any $Z' <_C Z$ (Attention: $\gamma_L$ provides this "correction" to $[L]$)

The T-variety $X \times X^A$ is simplectic w.r.t. $(\omega, -\omega|_{X^A})$.

**Proposition**

$\exists$ a T-invariant Lagrangian cycle $L_C$ on $X \times X^A$, proper over $X$, with the following properties:

(i) for any fixed connected component $Z$ of $X^A$, the restriction of $L_C$ to $X \times Z$ is supported on $\mathrm{Attr}_C^f(Z) \times Z$;

(ii) $L_C|_{Z \times Z} = \pm e(N_-(Z)) \cap [\Delta]$, according to the fixed polarization $\varepsilon$, where $\Delta$ = diagonal;

(iii) $\deg_A(L_C|_{Z' \times Z}) < \frac{1}{2}\mathrm{codim}_{\mathbb{R}}(Z')$ for any $Z' <_C Z$.

Proof of the existence of $\mathrm{Stab}_{C,\varepsilon}$:

$$\boxed{\mathrm{Stab}_C := \Theta_{L_C}}$$

## Proof of the Proposition:

Fix a connected component $Z$ of $X^A$ and set $\pm L_Z := [\overline{f^{-1}(\Delta)}]$, where (according to the polarization)

$$\boxed{\begin{aligned} f: \mathrm{Attr}_{\mathfrak{C}}(Z) \times Z &\longrightarrow Z \times Z \\ (x,y) &\longmapsto (\lim_{\mathfrak{C}}(x), y) \end{aligned}} \quad \text{and} \quad \boxed{\Delta \subset Z \times Z \text{ - diagonal}}$$

i.e.,

$$\boxed{f^{-1}(\Delta) = \bigcup_{y \in Z} \left(\mathrm{Attr}_{\mathfrak{C}}(\{y\}) \times \{y\}\right) \subset X \times X^A}$$

Fact: $\forall y \in Z$, $\mathrm{Attr}_{\mathfrak{C}}(\{y\})$ is Lagrangian - cf. [Hausel-Hitchin, arXiv:2101.08583 Proposition 2.10]

$\Downarrow$

Fact: $L_Z$ is a $A$-invariant Lagrangian cycle.

It is evident that $L_Z$ is supported on $\mathrm{Attr}^f_{\mathfrak{C}}(Z) \times Z$.
Moreover, one can check that:

$$L_Z|_{Z \times Z} = e(N_-(Z)) \cap [\Delta]$$

By applying the previous lemma, $\exists!\, \gamma_Z$ such that $L'_Z := L_Z + \gamma_Z$ satisfies (iii).
We define

$$\boxed{L_C := \sum_Z L'_Z}$$

We are left to prove that $\mathrm{supp}(L_C) \longrightarrow X$ is proper.

Let $X_0 \longrightarrow V$ be an $A$-equivariant embedding into a linear representation $V$ of $A$.
(see [Proposition 1.9, Brion, Introduction to actions of algebraic groups]).

Let $\sigma \in C$ and consider the induced $\mathbb{C}^*$-action on $V$. Set:

$$V_{\geq 0} := \mathrm{span}\{ v \in V : t \cdot v = t^m v \text{ for } m \geq 0 \} \quad \text{and} \quad V_0 = \{ v \in V : t \cdot v = v \}$$

Facts:

1. $V_{\geq 0} = V_0 \oplus V_{>0} \Rightarrow \exists\ \rho : V_{\geq 0} \longrightarrow V_0$ projection;
2. $\pi^{-1}(V_{\geq 0}) =: X_+ = \bigcup_Z \mathrm{Attr}_C(Z)$

By construction,
$$\boxed{\mathrm{Supp}(L_C) \subset X_+ \times_{V_0} X^A}$$

w.r.t. $\rho \circ \pi : \pi^{-1}(V_{\geq 0}) \longrightarrow V_0$.

We are left to prove that $X_+ \times_{V_0} X^A \longrightarrow X$ is proper. This is equivalent to prove that

$$X_+ \times_{V_0} X^A \longrightarrow X \xrightarrow{\pi} V$$

is proper, since $\pi$ is proper. The above composition is equivalent to

$$X_+ \times_{V_0} X^A \xrightarrow{\pi} V_{\geq 0} \times_{V_0} V_0 \longrightarrow V$$

Therefore we should prove that $V_{\geq 0} \times_{V_0} V_0 \longrightarrow V$ is proper: this is straightforward.

$\square$

Example

$X = T^* \mathbb{P}^1$. Consider

$$\boxed{\mathbb{C}^* \simeq A \subset T = A \times \mathbb{C}^*}$$

action on the fibers with weight $t_h$

For $t \in A \simeq \mathbb{C}^*$ and $[x, y] \in \mathbb{P}^1$, the A-action is:

$$t \cdot [x, y] = [t^n x, y]$$

We identify $\mathbb{P}^1$ with its zero section inside $X$. Then

$$X^A = \{p_0 = [1,0],\, p_1 = [0,1]\}$$

We have 2 roots:

$$\Delta = \{\alpha, -\alpha\} \subset \mathfrak{a}^*_{\mathbb{R}} \simeq \mathbb{R} \quad \text{with} \begin{cases} \alpha(t) = t^u \\ -\alpha(t) = t^{-u} \end{cases}$$

Then

$$\boxed{\mathfrak{a}_{\mathbb{R}} \simeq \mathbb{R} = C_+ \sqcup \alpha^\perp \sqcup C_-}$$

where

$$C_\pm = \{x \in \mathbb{R} : \langle \pm\alpha, x\rangle = u \cdot x > 0\} \quad \text{and} \quad \alpha^\perp = \{0\}$$

We have

$$\boxed{\mathrm{Attr}_{C_+}(p_0) = T^*_{p_0}\mathbb{P}^1} \quad \text{and} \quad \boxed{\mathrm{Attr}_{C_+}(p_1) = \mathbb{P}^1 \setminus \{p_0\}}$$

$$\boxed{\mathrm{Attr}_{C_-}(p_0) = \mathbb{P}^1 \setminus \{p_1\}} \quad \text{and} \quad \boxed{\mathrm{Attr}_{C_-}(p_1) = T^*_{p_1}\mathbb{P}^1}$$

In particular

$$\begin{array}{ccc} \overline{\mathrm{Attr}_{C_+}(p_1)} & & \overline{\mathrm{Attr}_{C_-}(p_0)} \\ \shortparallel & & \shortparallel \\ \mathbb{P}^1 \cap \{p_0\} \neq \phi & \text{and} & \mathbb{P}^1 \cap \{p_1\} \neq \phi \\ \Downarrow & & \Downarrow \\ \{p_0\} <_{C_+} \{p_1\} & \text{and} & \{p_1\} <_{C_-} \{p_0\} \end{array}$$

Therefore, finally we get

$$\boxed{\mathrm{Stab}_{C_+}(p_0) = -[T^*_{p_0}\mathbb{P}^1]} \quad \text{and} \quad \boxed{\mathrm{Stab}_{C_+}(p_1) = [\mathbb{P}^1] + [T^*_{p_0}\mathbb{P}^1]}$$

$$\boxed{\mathrm{Stab}_{C_-}(p_0) = [\mathbb{P}^1] + [T^*_{p_1}\mathbb{P}^1]} \quad \text{and} \quad \boxed{\mathrm{Stab}_{C_-}(p_1) = -[T^*_{p_1}\mathbb{P}^1]}$$

## R-matrix

Set $R := H^*_T(pt)$, $K := \mathrm{Frac}(R)$.

Fact: up to localization at $e(N_-(Z))$'s, $\mathrm{Stab}_C$ is invertible.

**Definition**

$$\boxed{R_{C',C} := \mathrm{Stab}_{C'}^{-1} \circ \mathrm{Stab}_C \in \mathrm{End}\left(H_T^*(X^A) \otimes_R K\right)}$$

**Example**

$X = T^*\mathbb{P}^1$, $\mathbb{C}^* \simeq A \subset T = A \times \mathbb{C}^*$ as before. As we saw before:

$$H_T^*(X^A) \simeq \mathbb{C}[u,\hbar]^{\oplus 2} \quad ; \quad H_T^*(X) \simeq \mathbb{C}[c,u,\hbar] / c(c-u)$$

with

$$[p_0] = \frac{c-u}{u} \quad \text{and} \quad [p_1] = -\frac{c}{u} \quad \in H_T^*(X) \otimes_R K$$

$$\mathrm{Stab}_{C_+}([p_0]) \underset{(ii)}{=} -e(N_-(\{p_0\})) \cup [p_0] = -u\left(\frac{c-u}{u}\right) = u - c$$

Similarly:

$$\mathrm{Stab}_{C_-}([p_1]) = -c \quad ; \quad \mathrm{Stab}_{C_+}([p_1]) = -c-\hbar \quad ; \quad \mathrm{Stab}_{C_-}([p_0]) = u-c-\hbar$$

Thus, in the basis $\{[p_0],[p_1]\}$, we have the matrices:

$$\mathrm{Stab}_{C_+} = \begin{pmatrix} -u & -\hbar \\ 0 & u-\hbar \end{pmatrix} \quad \text{and} \quad \mathrm{Stab}_{C_-} = \begin{pmatrix} -u-\hbar & 0 \\ -\hbar & u \end{pmatrix}$$

Thus

$$\mathrm{Stab}_{C_-}^{-1} = -\frac{1}{u(u+\hbar)} \begin{pmatrix} u & 0 \\ \hbar & -(u+\hbar) \end{pmatrix}$$

Therefore

$$R_{C_-,C_+} = \frac{1}{u+\hbar} \begin{pmatrix} u & \hbar \\ \hbar & u \end{pmatrix} \quad = \text{Yang R-matrix}$$