Fibrato cotangente

$$\mathcal{N} = \{ x \in \mathrm{Lie}(G) \mid x \text{ nilpotente} \}$$

e.g. $G = SL_2$ $\qquad \mathrm{Lie}(G) = \mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid d = -a \right\}$

$$\cup$$

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid -a^2 - bc = 0 \right\}$$

$$(\iff a^2 + bc = 0)$$

$\mathcal{N}$ è in generale $\mathbb{C}$-stabile $\Rightarrow$ "CONO NILPOTENTE"

$$\tilde{\mathcal{N}} = \{ (x, gB/B) \in \mathcal{N} \times \mathcal{F} \mid x \in \mathrm{Lie}(gBg^{-1}) \}$$

$G = SL_n$:

$$\tilde{\mathcal{N}} = \{ (x, gB/B) \in \mathcal{N} \times \mathcal{F} \mid x \in \mathrm{Lie}(gBg^{-1}) \}$$

$$= \{ (x, M_g = \{0\} \subset V_1 \subset \dots \subset V_{n-1} \subset \mathbb{C}^n) \mid x \in \mathrm{Lie}(\mathrm{Stab}\, M_g) \}$$

$$= \{ (x, M_g = \underset{\substack{\| \\ V_0}}{\{0\}} \subset V_1 \subset \dots \subset V_{n-1} \subset \underset{\substack{\| \\ V_n}}{\mathbb{C}^n} \mid x V_i \subseteq V_{i-1} \quad i = 1, \dots, n \}$$

e.g. $G = SL_2$

$$\tilde{\mathcal{N}} = \{ (x, [v]) \in \mathcal{N} \times \mathcal{F}l_n \mid x[v] \subseteq \{0\},\ x\mathbb{C}^2 \subseteq \mathbb{C}v \}$$

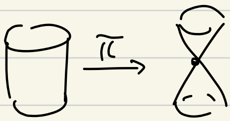
$x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $x = g \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g^{-1}$ $(\exists\, g \in SL_2)$

$\downarrow$

uccide ogni retta

$\rightsquigarrow$ esiste base $\langle v_1, v_2 \rangle \subseteq \mathbb{C}^2$ t.c. $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ $(x, [v_1])$ è l'unica coppia che va bene


$\xrightarrow{\pi}$

(π propria, fibre dette "fibre di Springer")

Andiamo a vedere l'altra proiezione

$$\begin{array}{ccl} \tilde{\mathcal{N}} & = & \{ (x, [v]) \in \mathcal{N} \times \mathbb{P}^1 \mid xv = 0 \quad x\mathbb{C}^2 \subseteq \mathbb{C}v \} \\ \varphi \downarrow & & \quad \downarrow \\ \mathbb{P}^1 & & \quad [v] \end{array}$$

$\varphi^{-1}[v]$ è una retta $\forall v$ $\rightsquigarrow$ fibrato!

(se $xv = 0$ $x\mathbb{C}^2 \subseteq \mathbb{C}v$ allora anche $(\lambda x v = 0)$ $\lambda x\, \mathbb{C}^2 \subseteq \mathbb{C}v$ $\forall \lambda \in \mathbb{C}$)

Andiamo a vedere che questo non è vero solo per SL_2, bensì $\tilde{\mathcal{N}} \xrightarrow{\varphi} \mathcal{F}$ è un fibrato, ed è proprio $T^*\mathcal{F}$!

Lemma (cf Chriss–Ginzburg, Lemma 1.4.9 +)

La proiezione $\tilde{\mathcal{N}} \xrightarrow{\varphi} \mathcal{F}$ identifica $(\tilde{\mathcal{N}}, \varphi)$ con il fibrato cotangente di $\mathcal{F}$.

Dim Denotiamo $\mathfrak{g} := Lie\, G$ e $\mathfrak{b} = Lie(B)$.

Lo spazio tangente in eB/B è $T_e(G/B) \simeq \mathfrak{g}/\mathfrak{b}$

Pertanto il suo duale è

$$T_e^*(G/B) = \{ f : \mathfrak{g} \to \mathbb{C} \mid f(\mathfrak{b}) = 0 \}$$

Il prodotto canonico $\langle \cdot , \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$

$$(x, y) \longmapsto Tr(xy)$$

è simmetrico e non-degenere $\Rightarrow$ ci permette di identificare $\mathfrak{g} \simeq \mathfrak{g}^*$

Otteniamo in questo modo:

$$T_e^*(G/B) \cong \{ x \in \mathfrak{g} \mid \langle x, \mathfrak{b} \rangle = 0 \} = \mathfrak{n} \left(= \text{radicale nilpotente di } \mathfrak{b} \right)$$

(per SL_n, GL_n, $\mathfrak{b} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$, $\mathfrak{n} = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}$)

$$T_{gB/B}(G/B) \cong \{ x \in \mathfrak{g} \mid \langle x, Lie(gBg^{-1}) \rangle \} = \mathfrak{n}(gBg^{-1})$$

Ma $\varphi^{-1}(gB/B) = \{ x \in \mathfrak{g} \mid x \in \mathcal{N} \wedge x \in Lie(gBg^{-1}) \} = \mathfrak{n}(gBg^{-1})$

□

L'identificazione del precedente risultato è G-equivariante.

$G \curvearrowright \tilde{\mathcal{N}}$ $\quad h.(x, gB/B) = (hxh^{-1}, (hg)B/B)$ $\quad h, g \in G$, $x \in \mathcal{N}$

$\rightsquigarrow$ azione del toro massimale $A \subseteq G$.

Punti fissi: cerchiamo coppie $(x, gB/B)$ t.c.

$$\begin{cases} hxh^{-1} = x \\ (hg)B/B = gB/B \end{cases} \quad \forall h \in A$$

gr. di Weyl

Ma
- $hgB/B = gB/B$ $\forall h \in A$ $\iff$ $\exists\, w \in W$ t.c. (cf lezione 2) $g \in \dot{w}B/B$
- $hxh^{-1} = x$ $\forall h \in A$ $\iff$ $x = 0$

$$\rightsquigarrow \tilde{\mathcal{N}}^A = \{(0, \dot{w}B/B) \mid w \in W\} \longleftrightarrow W$$

Vogliamo ora studiare i bacini attrattivi di tale azione (denotati sempre Attr da Francesco, "Leaf" da Su)

Richiamiamo notazione delle volte scorse:

$$\mathfrak{a}^*_{\mathbb{R}} := \mathrm{Hom}(\mathbb{C}^\times, A) \underset{\mathbb{Z}}{\otimes} \mathbb{R} \longrightarrow \mathbb{R}^r \quad (\text{se } A \simeq (\mathbb{C}^\times)^r)$$

$$\left(z \stackrel{\psi}{\longmapsto} (z^{e_1}, \dots, z^{e_r})\right) \otimes 1 \longmapsto (e_1, \dots, e_r)$$

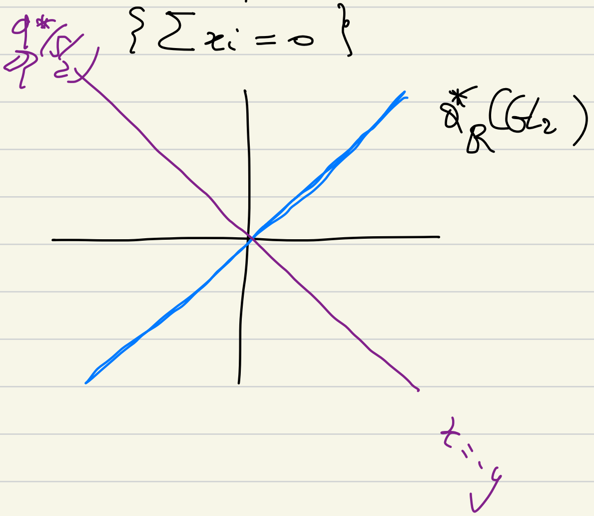
Camere: componenti connesse di $\mathfrak{a}^*_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Phi^+} \alpha^\perp$

e.g. GL_n $\quad \mathfrak{a}^*_{\mathbb{R}} \cong \mathbb{R}^n$ $\quad \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i t_j^{-1}$

camere comp. connesse di $\mathbb{R}^n \setminus \{x_i = x_j \mid i<j\}$

SL_n $\quad \mathfrak{a}^*_{\mathbb{R}}(SL_n) \hookrightarrow \mathfrak{a}^*_{\mathbb{R}}(GL_n)$

$$\mathfrak{a}^*_{\mathbb{R}}(SL_n) = \{\textstyle\sum x_i = 0\}$$



$\mathcal{C}$ camera (in $\mathfrak{a}^*_{\mathbb{R}}$), $\sigma \in \mathcal{C}$

$$\mathrm{Attr}_{\mathcal{C}}\left(\left(0, \dot{w}B/B\right)\right) = \left\{ \left(x, gB/B\right) \in \tilde{\mathcal{N}} \,\middle|\, \lim_{z \mapsto 0} \sigma(z) \cdot \left(x, gB/B\right) = \left(0, \dot{w}B/B\right) \right\}$$

$$= \left\{ \left(x, gB/B\right) \in \tilde{\mathcal{N}} \,\middle|\, \begin{array}{l} \lim_{z \mapsto 0} \sigma(z) x \sigma(z)^{-1} = 0 \\ \lim_{z \mapsto 0} (\sigma(z) g) B/B = \dot{w}B/B \end{array} \right\}$$

$$= \left\{ \left(x, gB/B\right) \in \tilde{\mathcal{N}} \,\middle|\, \begin{array}{l} gB/B \in \mathrm{Attr}_{\mathcal{C}}\left(\dot{w}B/B\right) \\ \lim_{z \mapsto 0} \sigma(z) x \sigma(z)^{-1} = 0 \end{array} \right\}$$

e.g. $G = SL_2$ $\qquad \chi : \mathbb{C}^\times \longrightarrow A$, $\quad z \longmapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ $\qquad \chi \in \mathcal{C}$

$$\mathrm{Attr}_{\mathcal{C}}\left(\left(0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right) = \left\{ \left(x, \begin{bmatrix} a \\ b \end{bmatrix}\right) \in \tilde{\mathcal{N}} \,\middle|\, \begin{array}{l} \begin{bmatrix} a \\ b \end{bmatrix} \in \mathrm{Attr}_{\mathcal{C}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \\ \lim_{z \mapsto 0} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} x \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} = 0 \end{array} \right\}$$

$$= \left\{ \left(\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \,\middle|\, u \in \mathbb{C} \wedge \lim_{z \mapsto 0} \underbrace{\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}}_{\begin{pmatrix} 0 & z^2 u \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$= \left\{ \left(\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \,\middle|\, u \in \mathbb{C} \right\} = \mathrm{Leaf}_{\mathcal{C}}\left(0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$\mathrm{Attr}_e\left(\left(0,\begin{bmatrix}0\\1\end{bmatrix}\right)\right)=\left\{\left(\begin{pmatrix}ue & -ue^2\\ u & -ue\end{pmatrix},\begin{bmatrix}e\\1\end{bmatrix}\right)\ \middle|\ \begin{matrix}u\in\mathbb{C}\\ \begin{pmatrix}ue & -z^2ue^2\\ z^{-2}u & -ue\end{pmatrix}\xrightarrow[z\to 0]{}\begin{pmatrix}0&0\\0&0\end{pmatrix}\end{matrix}\right\}$$

$$=\left\{\left(\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{bmatrix}e\\1\end{bmatrix}\right)\ \middle|\ e\in\mathbb{C}\right\}$$

In generale:

$$\begin{matrix}G & \rightsquigarrow & \mathfrak{g}=\mathrm{Lie}(G)\supset\mathfrak{b}\\ \cup & & \\ B\supset A & & =\bigoplus_{\alpha\in\mathrm{Hom}(T,\mathbb{C}^\times)}\mathfrak{g}_\alpha\\ (\Delta)\ (d^0) & & \end{matrix}\qquad \mathfrak{g}_\alpha=\{x\in\mathfrak{g}\mid txt^{-1}=\alpha(t)x\}$$

$$\{\alpha\mid\alpha\neq(t\mapsto 1),\ \mathfrak{g}_\alpha\subset\mathfrak{b}\}=\Phi^+$$

$$\{\alpha\mid\alpha\neq(t\mapsto 1)\}=\Phi$$

e.g. $$\mathfrak{sl}_2=\mathbb{C}\begin{pmatrix}0&1\\0&0\end{pmatrix}\oplus\mathbb{C}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\oplus\mathbb{C}\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

$$\alpha:\begin{pmatrix}z&0\\0&z^{-1}\end{pmatrix}\mapsto z^2\qquad \begin{pmatrix}z&0\\0&z^{-1}\end{pmatrix}\mapsto 1\qquad -\alpha:\begin{pmatrix}z&0\\0&z^{-1}\end{pmatrix}\mapsto z^{-2}$$

Per $\mathcal{C}$ corrispondente a decomp. di Bruhat:

$$\mathrm{Attr}_{\mathcal{C}}\left(\left(0, \dot{\omega} B/B\right)\right) = \left\{ \left(x, gB/B\right) \;\middle|\; \begin{matrix} g \in B\dot{\omega}B \\ x \in \bigoplus_{\substack{\alpha \in \Phi^+ \\ \omega\alpha > 0}} \mathfrak{g}_\alpha \end{matrix} \right\}$$

<u>Rmk</u> Tale bacino attrattivo è iso al fibrato conormale alla varietà $B\dot{\omega}B/B$

<u>e.g.</u> SL_2:

$$\mathrm{Attr}_{\mathcal{C}}\Big(\big(0, \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{= \mathrm{Id}\, B/B}\big)\Big) = \left\{ \left(\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \;\middle|\; u \in \mathbb{C} \right\}$$

$$= \mathfrak{g}_\alpha \times \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$\mathrm{Id}\,\alpha = \alpha > 0$

$$\mathrm{Attr}_{\mathcal{C}}\Big(\big(0, \underbrace{\begin{bmatrix} a \\ 1 \end{bmatrix}}_{= (12)B/B}\big)\Big) = \left\{ \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{bmatrix} a \\ 1 \end{bmatrix} \right) \;\middle|\; a \in \mathbb{C} \right\}$$

$\Phi^+ = \{\alpha\}$ $\qquad (12)\alpha = -\alpha$

Ordinamento parziale sui punti fissi

$$(\mathcal{F}^A, \leq_{\mathfrak{C}})$$

$$\hookrightarrow w_1 \leq_{\mathfrak{C}} w_2 \iff \mathrm{Attr}_{\mathfrak{C}}(w_1 B/B) \cap \overline{\mathrm{Attr}_{\mathfrak{C}}(w_2 B/B)} \neq \emptyset$$

(nel caso delle varietà di bandiera

$$\mathrm{Attr}_{\mathfrak{C}}(\dot{w}_1 B/B) \cap \overline{\mathrm{Attr}_{\mathfrak{C}}(\dot{w}_2 B/B)} \neq \emptyset$$

$$\Longrightarrow \mathrm{Attr}_{\mathfrak{C}}(\dot{w}_1 B/B) \subseteq \overline{\mathrm{Attr}_{\mathfrak{C}}(\dot{w}_2 B/B)} \quad)$$

Ricordiamo che tale ordinamento svolgeva un ruolo fondam. nella definizione della stable envelope e pertanto dobbiamo capirlo per bene:

e.g. SL_2 $\quad \chi : z \longmapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$

$$\mathrm{Attr}_{\mathfrak{C}}\left(0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \left\{ \left(\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \middle| u \in \mathbb{C} \right\} \text{ è chiuso}$$

$$\overline{\mathrm{Attr}_{\mathfrak{C}}\left(\left(0, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right)} = \overline{\left\{ \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{bmatrix} e \\ 1 \end{bmatrix} \right) \middle| e \in \mathbb{C} \right\}}$$

$$= \left\{ \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right) \middle| \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{P}^1 \right\}$$

$$\Longrightarrow \left(0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \leq_{\mathfrak{C}} \left(0, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Infatti, la parte nilpotente è sempre chiusa e la chiusura "avviene sulla parte di bandiera".

Otteniamo allora un isomorfismo di insiemi parzialmente ordinati:

$$\left((\tilde{\mathcal{N}})^A, \leq_{\mathfrak{e}} \right) \xrightarrow{\sim} \left(\mathcal{F}^A, \leq_{\mathfrak{e}} \right)$$

$$\left(\underline{0}, \dot{w}B/B \right) \longmapsto \dot{w}B/B$$

Ricordiamo, dalla lezione 4 di Francesco, le sottovarietà chiuse

$$\mathrm{Attr}^{f}_{\mathfrak{e}}\left((0, \dot{w}B/B) \right) = \bigcup_{\substack{y \in W \\ y \leq_{\mathfrak{e}} w}} \mathrm{Attr}_{\mathfrak{e}}\left((0, \dot{y}B/B) \right) \left(= \mathrm{Slope}_{\mathfrak{e}}(w) \text{ in } S_u \right)$$