Thm [Su] Sia $\mathcal{C} = \mathcal{C}^+$ (= camera corrispondente a dcp di Bruhat)

$\exists!$ morfismo di $R = H^*_T(pt)$ - moduli :

$$\mathrm{Stab}_+ : \bigoplus_{w \in W} R \longrightarrow H^*_T(\tilde{\mathcal{N}}) \xrightarrow{\iota^*} \bigoplus_{w \in W} R$$

$\cong H^*_T(\tilde{\mathcal{N}}^A)$

t.c.

Bruhat

1) $\iota^* \big(\mathrm{stab}_+ ((0, \dots, \overset{w}{1}, 0, \dots 0)\big)_y = 0 \quad \forall\, y \nleq w$

2)
$$\iota^* (\mathrm{stab}_+ (0, \dots 0, \overset{w}{1}, 0, \dots 0))_w = \prod_{\substack{\alpha \in \Phi^- \\ w^{-1}\alpha \in \Phi^+}} (\alpha - \hbar) \prod_{\substack{\beta \in \Phi^+ \\ w^{-1}\beta \in \Phi^+}} \beta$$

3) $\forall$ radice semplice α t.c. $\ell(ws_\alpha) = \ell(w) + 1$:

$$(\iota^* \mathrm{stab}_+ (0, \dots, 0, \overset{ws_\alpha}{1}, 0, \dots 0))_y =$$
$$- \frac{\hbar}{y\alpha} (\iota^* \mathrm{stab}_+ (0, \dots, 0, \overset{w}{1}, 0, \dots, 0))_y - \frac{y\alpha - \hbar}{y\alpha} (\iota^* \mathrm{stab}_+ (0, \dots, 0, \overset{w}{1}, 0, \dots, 0))_{ys_\alpha}$$

e.g. $G = SL_2$ $\quad \iota^* \mathrm{stab}_+ ((\overset{s}{0}, \overset{e}{1})) = (0, \alpha)$ $\quad \varepsilon_1 - \varepsilon_2$

$\quad \iota^* \mathrm{stab}_+ ((\overset{s}{1}, \overset{e}{0})) = (-\alpha - \hbar, -\hbar)$

$w = e \quad y = e \quad -\frac{\hbar}{\cancel{\alpha}} \cancel{\alpha} + 0$

Thm [Su] Sia $\mathcal{C} = \mathcal{C}^{+w_0} = \mathcal{C}^-$ (= camera opposta a $\mathcal{C}^+$)

$\exists!$ morfismo di $R = H_T^*(pt)$ - moduli:

$$\mathrm{stab}_- : \bigoplus_{w \in W} R \longrightarrow H_T^*(\tilde{\mathcal{N}})$$

t.c.

1) $\iota^*(\mathrm{stab}_-((0,\dots,\overset{w}{1},0,\dots 0)))_y = 0 \quad \forall y \not\geq w$

2) $$\iota^*(\mathrm{stab}_-(0,\dots 0,\overset{w}{1},0,\dots 0))_w = \prod_{\substack{\alpha \in \Phi^+ \\ w^{-1}\alpha \in \Phi^+}} (\alpha - \hbar) \prod_{\substack{\beta \in \Phi^- \\ w^{-1}\beta \in \Phi^+}} \beta$$

3) $\forall$ radice semplice α t.c. $\ell(ys_\alpha) = \ell(y)+1$:

$$(\iota^*\mathrm{stab}_-(0,\dots,0,\overset{w}{1},0,\dots 0))_{ys_\alpha} =$$

$$-\frac{\hbar}{y\alpha - \hbar}(\iota^*\mathrm{stab}_-(0,\dots,0,\overset{w}{1},0,\dots,0))_y - \frac{y\alpha}{y\alpha - \hbar}(\iota^*\mathrm{stab}_-(0,\dots,0,\overset{ws_\alpha}{1},0,\dots,0))_y$$

e.g. $G = SL_2$ $\qquad \iota^*\mathrm{stab}_-((0,1)) = (-\hbar, \alpha - \hbar)$

$\qquad \iota^*\mathrm{stab}_-((1,0)) = (-\alpha, 0)$

Dim · unicità ✓

· $\exists$ per induzione ; base ✓ $\quad \mathrm{stab}_-(w_0)$ $\quad \mathrm{stab}_+(e)$

Passo induttivo tramite corrisp.:

$$G \curvearrowright \mathcal{F} \times \mathcal{F} = G/B \times G/B \qquad h.(g_1 B/B, g_2 B/B) = (hg_1 B/B, hg_2 B/B)$$

$$\begin{tikzcd} & \mathcal{F}\times\mathcal{F} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{F} & & \mathcal{F} \end{tikzcd}$$

$\leftarrow$ G-eqvt

$\Longrightarrow$ $\overline{G\text{-orbite}}$ su $\mathcal{F} \times \mathcal{F} \longleftrightarrow W$

Infatti: $\forall\, \mathcal{Y}$ $\overline{G\text{-orbite}}$

$$p_1(\mathcal{Y} \cap \mathcal{F} \times \{B/B\}) = \overline{BwB}/B \quad \exists\, w \in W$$

(p.e. | $\boxed{w=e}$) $\quad p_1(\mathcal{Y}_e \cap \mathcal{F} \times \{B/B\}) = \overline{BeB}/B = B/B$

$$\mathcal{Y}_e = \{(gB/B, gB/B) \mid g \in G\}$$

Vogliamo considerare Y_{s_α} : $\overline{G\text{-orbita}}$ in $\mathcal{F}\times\mathcal{F}$ corrisp. a s_α

$$\left(\text{cioè}: p_1\left(Y_{s_\alpha}\cap \mathcal{F}\times\{B/B\}\right) = \overline{Bs_\alpha B}/B\right)$$

$$\begin{tikzcd} T^*_{Y_{s_\alpha}}(\mathcal{F}\times\mathcal{F}) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & T^*\mathcal{F} \\ T^*\mathcal{F} & \end{tikzcd}$$

$$D_\alpha := \pi_{1*}\circ\pi_2^* : H^*_T(\underset{\underset{T^*\mathcal{F}}{\|}}{\tilde{N}}) \to H^*_T(\underset{\underset{T^*\mathcal{F}}{\|}}{\tilde{N}})$$

<u>Propn</u> (α semplice)

$$\begin{tikzcd} H^*_T(\tilde{N}) \arrow[r, "D_\alpha"] \arrow[d, "\iota^*"] & H^*_T(\tilde{N}) \arrow[d, "\iota^*"] \\ \mathrm{Im}(\iota^*) \arrow[r, "A_\alpha"] & \mathrm{Im}(\iota^*) \end{tikzcd}$$

dove $A_\alpha((f_y)_{y\in W}) = (f'_y)_{y\in W}$ con $f'_y = \dfrac{f_{ys_\alpha} - f_y}{y\alpha}(y\alpha - \hbar)$

<u>Lemma</u> se $ws_\alpha > w$, allora

$$D_\alpha \underbrace{\mathrm{stab}_\pm(0,\_,\overset{w}{1},0\_0)}_{\mathrm{stab}_\pm(w)} = -\mathrm{stab}_\pm(w) - \mathrm{stab}_\pm(ws_\alpha)$$

dove $\quad A_\alpha\left((f_y)_{y\in W}\right) = (f'_y)_{y\in W}$ con $f'_y = \dfrac{f_{ys_\alpha} - f_y}{y\alpha}(y\alpha - \hbar)$

$$D_\alpha \underbrace{\mathrm{stab}_\pm(0,\dots,\overset{w}{1},0\dots0)}_{\mathrm{stab}_\pm(w)} = -\mathrm{stab}_\pm(w) - \mathrm{stab}_\pm(ws_\alpha)$$

$$\begin{array}{ccc} & 321 & \\ 231 & & 312 \\ s_\alpha = 213 & & 132 \\ & 123 = e & \end{array}$$

e.g. $G = SL_3$

$$\iota^* \mathrm{stab}_+(e) = \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ 0 & & 0 \\ & \alpha\beta(\alpha+\beta) & \end{array}$$

$\alpha = \varepsilon_1 - \varepsilon_2$

$$\iota^* \mathrm{stab}_+(s_\alpha) = \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ 0 & & 0 \\ & -\alpha\beta(\alpha+\beta) & \end{array} - \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ (\alpha+\beta)\beta(-\alpha-\hbar) & & 0 \\ & -(\alpha+\beta)\beta(\alpha-\hbar) & \end{array} = \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ (\alpha+\beta)\beta(-\alpha-\hbar) & & 0 \\ & (\alpha+\beta)\beta(-\hbar) & \end{array}$$

$$A_\alpha\left(\begin{array}{ccc} & 0 & \\ 0 & & 0 \\ 0 & & 0 \\ & \alpha(\alpha+\beta)\beta & \end{array}\right)_e = \frac{-\cancel{\alpha}(\alpha+\beta)\beta}{\cancel{\alpha}}(\alpha-\hbar)$$

$$A_\alpha\left(\begin{array}{ccc} & 0 & \\ 0 & & 0 \\ 0 & & 0 \\ & \alpha(\alpha+\beta)\beta & \end{array}\right)_{s_\alpha} = \frac{\cancel{\alpha}(\alpha+\beta)\beta}{-\cancel{\alpha}}(-\alpha-\hbar)$$

$$\mathcal{F}, \mathfrak{C} \rightsquigarrow \left\{ \mathrm{stab}_{\mathfrak{C}}\big(\overset{(0,\dots,0,1,0,\dots,0)}{w}\big) \right\}_{w \in W}$$

$$\Big\updownarrow$$

$$\mathrm{Stab}_{\mathfrak{C}} = \left(\left({}^{*}\mathrm{stab}_{\mathfrak{C}}(w) \right)^{T} \right)_{w \in W} \qquad \text{(fissato ordinamento totale su } W\text{)}$$

invertibile in $\mathrm{Mat}_{\#W}\left(H_T^*(pt)\left[e(N_{-,x}) \mid x \in W \right] \right)$

$\forall\, \mathfrak{C}, \mathfrak{C}'$

$$\begin{tikzcd}
 & H_T^*(\tilde{\mathcal{N}}) \otimes \mathbb{Q} \arrow[dr, "\mathrm{stab}_{\mathfrak{C}'}"] & \\
H_T^*(\tilde{\mathcal{N}}^A) \otimes \mathbb{Q} \arrow[ur, "\mathrm{stab}_{\mathfrak{C}}"] \arrow[rr, "R_{\mathfrak{C}',\mathfrak{C}}"'] & & H_T^*(\tilde{\mathcal{N}}^A) \otimes \mathbb{Q}
\end{tikzcd}$$

R- matrice geometrica

e.g. $\mathbb{P}^1$; $\quad \mathrm{Stab}_+ = \begin{pmatrix} \alpha & -\hbar \\ 0 & -\alpha-\hbar \end{pmatrix} \qquad \mathrm{stab}_- = \begin{pmatrix} \alpha-\hbar & 0 \\ -\hbar & -\alpha \end{pmatrix}$

$$\mathrm{stab}_-^{-1} = \frac{1}{-\alpha(\alpha-\hbar)} \begin{pmatrix} -\alpha & 0 \\ \hbar & \alpha-\hbar \end{pmatrix}$$

e.g. $\mathbb{P}^1$;

$$\mathrm{stab}_+ = \begin{pmatrix} \alpha & -\hbar \\ 0 & -\alpha-\hbar \end{pmatrix} \qquad \mathrm{stab}_- = \begin{pmatrix} \alpha-\hbar & 0 \\ -\hbar & -\alpha \end{pmatrix}$$

$$\mathrm{stab}_-^{-1} = \frac{1}{-\alpha(\alpha-\hbar)} \begin{pmatrix} -\alpha & 0 \\ \hbar & \alpha-\hbar \end{pmatrix}$$

$$R_{\epsilon^-,\epsilon^+} = \begin{pmatrix} \frac{\alpha}{\alpha-\hbar} & \frac{-\hbar}{\alpha-\hbar} \\ \frac{-\hbar}{\alpha-\hbar} & \frac{\alpha}{\alpha-\hbar} \end{pmatrix} = \frac{1}{\alpha-\hbar}\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & -\hbar \\ -\hbar & 0 \end{pmatrix} \right)$$

$$H_T^*(\overbrace{T^*\mathbb{P}^1}^{\left(0,\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right)\ \left(0,\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right)}{}^A) \otimes_{\mathbb{C}[\alpha,\hbar]} \mathbb{C}(\alpha,\hbar) \longrightarrow H_T^*(T^*\mathbb{P}^1{}^A) \otimes \mathbb{C}(\alpha,\hbar)$$

$$\cong$$

$$\mathbb{C}[\alpha,\hbar]^{\oplus 2} \otimes_{\mathbb{C}[\alpha,\hbar]} \mathbb{C}(\alpha,\hbar)$$

$$\cong$$

$$\mathbb{C}^{\oplus 2} \otimes_{\mathbb{C}} \mathbb{C}(\alpha,\hbar)$$

Varietà di bandiere parziali

Fissiamo $N, n \in \mathbb{Z}_{\geq 1}$ $\quad \lambda \vdash n$ $\quad \lambda \in \mathbb{Z}_{\geq 0}^N$, $\lambda = (\lambda_1, \dots, \lambda_N)$

$\boxed{A}$ $\rightsquigarrow$ $$\mathcal{F}_\lambda = \left\{ \{0\} \subset V_1 \subset \dots \subset V_N = \mathbb{C}^n \,\middle|\, \dim V_i / V_{i-1} = \lambda_i \right\}$$

$N-1$

$$\mathcal{X}_n = \bigsqcup_{\lambda \vdash n} T^* \mathcal{F}_\lambda$$

e.g. $\boxed{n=1}$ $\quad \lambda = \underbrace{(0, \dots, 0, \overset{i}{1}, 0, \dots, 0)}_{N}$

$$\Rightarrow \mathcal{F}_\lambda = pt \Rightarrow T^*\mathcal{F}_\lambda = pt$$

$$\mathcal{X}_n = \{x_1, \dots, x_N\}$$

$\boxed{n=2}$ $\quad \lambda = (0, \dots 0, \overset{i}{1}, 0, \text{—} 0, \overset{j}{1}, 0 \text{—} 0)$ $\qquad \lambda = (0, \text{—}, 0, 2, 0, \text{—} 0)$

$$\mathcal{F}_\lambda \simeq \mathbb{P}^1 \qquad\qquad \mathcal{F}_\lambda = pt$$

$$\Rightarrow T^*\mathbb{P}^1 = T^*\mathcal{F}_\lambda \qquad\qquad T^*\mathcal{F}_\lambda = pt$$

e.g. $\boxed{n=1}$ $$H_T^*(\underset{\{x_1 \text{—} x_N\}}{\mathcal{X}_1^{A \simeq \mathbb{C}^\times}}) = \bigoplus_{i=1}^N \mathbb{C}[\alpha, \hbar] \simeq \mathbb{C}^N \otimes \mathbb{C}[\alpha, \hbar]$$

$\boxed{n=2}$ $\quad H_T^*\left(\mathcal{X}_2^{A=(\mathbb{C}^*)^2}\right) = \mathbb{C}[\alpha,\hbar]^{\oplus N^2} \cong \left(\mathbb{C}^{\boxed{N}}\right)^{\otimes 2} \otimes \mathbb{C}[\alpha,\hbar]$

N^2 punti fissi: $2\cdot\frac{N\,(N-1)}{2}$ (da $T^*\mathbb{P}^1$)

N (da T^*pt)

$$R_{\varepsilon^-,\varepsilon^+} = \begin{pmatrix} \frac{\alpha}{\alpha-\hbar} & \frac{-\hbar}{\alpha-\hbar} \\ \frac{-\hbar}{\alpha-\hbar} & \frac{\alpha}{\alpha-\hbar} \end{pmatrix} = \frac{1}{\alpha-\hbar}\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & -\hbar \\ -\hbar & 0 \end{pmatrix}\right)$$

$-\hbar\cdot$(flip $\mathbb{C}^N \otimes \mathbb{C}^N$)